A Topological Characterization of Modulo-p Arguments and Implications for Necklace Splitting

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Abstract

The classes PPA-p have attracted attention lately, because they are the main candidates for capturing the complexity of *Necklace Splitting* with p *thieves*, for prime p. However, these classes were not known to have complete problems of a topological nature, which impedes any progress towards settling the complexity of the Necklace Splitting problem. On the contrary, topological problems have been pivotal in obtaining completeness results for PPAD and PPA, such as the PPAD-completeness of finding a Nash equilibrium [Daskalakis et al., 2009, Chen et al., 2009b] and the PPA-completeness of Necklace Splitting with 2 *thieves* [Filos-Ratsikas and Goldberg, 2019].

In this paper, we provide the first *topological characterization* of the classes PPA-p. First, we show that the computational problem associated with a simple generalization of Tucker's Lemma, termed p-POLYGON-TUCKER, as well as the associated Borsuk-Ulam-type theorem, p-POLYGON-BORSUK-ULAM, are PPA-p-complete. Then, we show that the computational version of the well-known *BSS Theorem* [Bárány, Shlosman, and Szücs, 1981], as well as the associated BSS-TUCKER problem are PPA-p-complete. Finally, using a different generalization of Tucker's Lemma (termed $\mathbb{Z}_p$-STAR-TUCKER), which we prove to be PPA-p-complete, we prove that p-thief Necklace Splitting is in PPA-p. This latter result gives a new combinatorial proof for the Necklace Splitting theorem, the only proof of this nature other than that of Meunier [2014].

All of our containment results are obtained through a new combinatorial proof for $\mathbb{Z}_p$-versions of Tucker's lemma that is a natural generalization of the standard combinatorial proof of Tucker's lemma by Freund and Todd [1981]. We believe that this new proof technique is of independent interest.

Contents

1 Introduction

The class TFNP [Megiddo and Papadimitriou, 1991] is the class of *Total Search Problems in NP*, i.e., problems for which a solution is always guaranteed to exist, and can be verified in polynomial time. In a seminal paper, attempting to capture the complexity of numerous interesting problems, Papadimitriou [1994] defined several subclasses of TFNP, such as PPAD, PPA and PPP, among others, each of which is associated with a different existence principle. For example, PPAD is based on the principle that given a source in a directed graph with in-degree and out-degree at most 1, there must exist another vertex of degree 1; PPA is based on a similar principle on an undirected graph, and PPP is based on the pigeonhole principle.

Among those, PPAD has been largely successful in capturing the complexity of many well-known problems, most prominently that of computing Nash equilibria in games [Daskalakis et al., 2009, Chen et al., 2009b]. PPA and PPP have been more elusive in that regard for nearly two decades, until the recent results of Filos-Ratsikas and Goldberg [2018, 2019] and Sotiraki et al. [2018], who provided the first "natural" complete problems for these classes respectively. Here, a "natural" problem is one that does not explicitly include a polynomial-sized circuit in its definition (as termed in [Grigni, 2001]). Interestingly, the PPA-complete problems in [Filos-Ratsikas and Goldberg, 2018, 2019] that solidified the status of PPA as a class containing such natural problems were the well-known *Necklace Splitting* problem for *two thieves* as well as its continuous variant (coined the *Consensus-Halving* problem in [Simmons and Su, 2003]).

The Necklace Splitting problem is a classical problem in combinatorics, dating back to the mid-1980s and the works of Goldberg and West [1985], Alon and West [1986] and Alon [1987], among others. In this problem, k thieves are aiming to split an open necklace containing n beads of t different colors (with exactly $k \cdot a_i$ beads of color i, for some $a_i \in \mathbb{N}$), such that each thief receives exactly a_i beads of color i. Furthermore, the thieves are allowed to use only $(k-1)t$ cuts to obtain this division. A solution to the Necklace Splitting problem is guaranteed to exist, as was proven by Alon [1987]. Earlier on, Goldberg and West [1985] and Alon and West [1986] had proven the existence of a solution for the case of 2 thieves; this result invokes a fundamental tool from mathematics, the *Borsuk-Ulam Theorem* [Borsuk, 1933]. Alon's proof for the general case proceeds in two steps. First, using a simple argument, he proves that if the theorem holds for any prime number p of thieves, it also holds for any other number of thieves. In the second step, which is significantly more involved, he proves the theorem for any prime number by using the *BSS Theorem*, a generalization of the Borsuk-Ulam Theorem due to Bárány, Shlosman, and Szücs [1981].

Questions about the computational complexity of finding a solution were raised explicitly as early as when the first existence results were proven [Goldberg and West, 1985] and then later on by a series of papers [Alon, 1988, 1990, Meunier, 2008, Meunier and Sebő, 2009, Meunier and Neveu, 2012]. The first definitive answer was provided by Filos-Ratsikas and Goldberg [2019], via their PPA-completeness result, following an initial PPAD-hardness result by Filos-Ratsikas et al. [2018]. Crucially however, their result only applies to the case of two thieves.[1] In fact, the authors observed (also referencing de Longueville and Živaljević [2006] and Meunier [2014] as previously having made similar observations) that the version of the problem with $p \geq 3$ thieves does not seem to boil down to the principle associated with the class PPA. To this end, they conjectured that p-thief Necklace Splitting is complete for the computational class PPA-p, also defined by

[1]The authors also extend the PPA membership straightforwardly to numbers of thieves which are powers of 2.

Papadimitriou [1994], in which the associated principle is the following; Given a vertex with degree which is not a multiple of p in a bipartite graph, find another such vertex. It follows from the definition that PPA-2 = PPA.

The classes PPA-p have been very recently studied by Hollender [2019] and Göös et al. [2020]. The authors of the latter paper in fact provide the first PPA-p-completeness result for a natural problem, a computational version of the Chevalley-Warning theorem. Importantly, they were able to obtain their completeness result via reductions to and from equivalent variants of the canonical problems of the class. To prove any results about p-thief Necklace Splitting however, such an approach seems insufficient.

To see this, note that the results for $p = 2$, both for the PPA-membership [Filos-Ratsikas et al., 2018] and for PPA-hardness [Filos-Ratsikas and Goldberg, 2019] of the problem, are obtained via reductions to/from a computational version of *Tucker's Lemma* [Tucker, 1945], a discrete analogue to the Borsuk-Ulam theorem, proven to be PPA-complete by Papadimitriou [1994] and Aisenberg et al. [2020]. Tucker's lemma asserts that if we have an antipodally symmetric triangulation of a d-dimensional ball B and a labeling function which assigns complementary labels to antipodal points on the boundary, then there is a *complementary edge*, i.e., two adjacent points with equal-and-opposite labels. This connection is not a coincidence; for example, the idea in [Filos-Ratsikas et al., 2018] is in fact a direct adaptation of a combinatorial existence proof of Simmons and Su [2003] for the Consensus-Halving problem, which goes via Tucker's lemma.

In order to obtain a PPA-p result for p-thief Necklace Splitting, it seems rather imperative to develop an "arsenal" of computational problems of a related nature, that we could reduce to/from, namely generalizations of the Borsuk-Ulam theorem and Tucker's lemma. Such "topological characterizations" did not only enable researchers in settling the complexity of the problem (and some other related problems) for PPA = PPA-2, but also facilitated the success of PPAD to the utmost extent, seeing as virtually all the related important results go via the computational versions of the associated topological theorems (specifically *Brouwer's Fixed-Point Theorem* [Brouwer, 1911] and *Sperner's Lemma* [Sperner, 1928]).[2] In a very related manner, Simmons and Su [2003] stressed the importance of obtaining such a generalization, in the quest for obtaining a combinatorial proof of *Consensus-*$1/k$*-Division* (the generalization of Consensus-Halving) and therefore, for Necklace Splitting with k thieves, for $k > 2$. Lastly, Göös et al. [2020] explicitly raised the complexity of one of these generalizations, the BSS Theorem, as an open problem.

1.1 Our Results

In this paper, we obtain such a *topological characterization* of the classes PPA-p, for all primes $p \geq 3$. Namely, we provide generalizations of the computational versions of the Borsuk-Ulam theorem and Tucker's lemma, parameterized by p, which are complete for PPA-p. A highlight of our generalizations is the PPA-p completeness of the computational version of the BSS Theorem. Finally, we use a further generalization to prove that Necklace Splitting with p thieves lies in PPA-p.

The strength of our results lies in that

- ⋄ they solidify the status of the classes PPA-p as classes containing interesting well-known problems (adding to the recent results of Göös et al. [2020]), and

[2] To be more precise, several important PPAD-hardness results were obtained via reductions from the *Generalized Circuit problem* [Daskalakis et al., 2009, Chen et al., 2009b, Rubinstein, 2018], which was however proven to be PPAD-complete via the aforementioned topological problems.

- ◇ they set up an essential toolkit for obtaining more completeness results for the classes, e.g., possibly the PPA-p-completeness of p-thief Necklace Splitting.

All of our PPA-p-membership results are obtained via a new combinatorial proof for $\mathbb{Z}_p$-versions of Tucker's lemma. This new proof can be seen as a natural generalization of the standard combinatorial proof of Tucker's lemma by Freund and Todd [1981]. Thus, as a byproduct of our techniques we also obtain the following results:

1. A combinatorial proof of the BSS theorem. The original proof by Bárány et al. [1981] is not combinatorial, as it uses various tools from algebraic topology. Using our new technique, we are able to provide the first combinatorial proof for this theorem.

2. A combinatorial proof of the Necklace Splitting theorem. The existence of such a combinatorial proof had been an open problem since [Alon, 1987]. This open problem was solved by Meunier [2014] using a rather complicated argument. In contrast, our new combinatorial proof uses more elementary tools and is conceptually simpler than Meunier's proof.

3. A stronger statement of the continuous Necklace Splitting theorem which is called Consensus-$1/p$-Division [Alon, 1987, Simmons and Su, 2003]. The main advantage of our new theorem is that it actually works for valuation functions that are not necessarily additive and non-negative, for details see Theorem 6.5.

As a result, we believe that this new technique is of independent interest and will be useful for providing combinatorial proofs of other topological existence theorems such as Dold's Theorem Dold [1983]. We remark here that although the original proof of the Necklace Splitting theorem in [Alon, 1987] is via the BSS Theorem, our PPA-p membership result for Necklace Splitting does *not* go via the PPA-p membership result that we prove for BSS. This is due to the fact that for several steps of the proof of Alon [1987], it is quite unclear whether they can be carried out in polynomial time. Instead, we construct a reduction directly from $\mathbb{Z}_p$-STAR-TUCKER, which we prove to be PPA-p-complete.

In the remainder of this section, we informally state our main problems and results, and we give a short and high-level description of our proof techniques. We start with a topological theorem that is easy to state in Section 1.1.1, which we call k-Polygon Tucker's Lemma. Then, we present our results about the completeness of the well-studied BSS Theorem in Section 1.1.2. Finally, we briefly explain how we get our new combinatorial proof of Necklace Splitting and the containment in PPA-p, in Section 1.1.3.

Throughout this paper, unless otherwise specified, k denotes an integer larger or equal to 2, and p denotes a prime number.

1.1.1 The k-Polygon Borsuk-Ulam Theorem

The k-Polygon Borsuk-Ulam theorem can be understood via the corresponding statement of Borsuk-Ulam in 2 dimensions. First, let us introduce the following notion of equivariance for a function. Let S^1 be the unit circle in 2-dimensions and B^2 be the unit disk. We say that a function $g : S^1 \to \mathbb{R}^2$ is *equivariant to a rotation of a° degrees* if whenever we rotate the input $\boldsymbol{x}$ by a°, the image $g(\boldsymbol{x})$ is also rotated by a° degrees. We extend this definition to functions $f : B^2 \to \mathbb{R}^2$ and we say that f is *equivariant to a rotation of a° degrees on the boundary* if the restriction of f to the boundary S^1 is equivariant to a rotation of a° degrees. Using this language, the following is one of the many equivalent ways to state the classical Borsuk-Ulam Theorem in 2 dimensions (see the book by Matoušek [2008] for various equivalent versions).

Informal Theorem 1 (2D Borsuk-Ulam Theorem). *Let $f : B^2 \to \mathbb{R}^2$ be a continuous function that is equivariant to a rotation of* $180°$ *degrees on the boundary. Then, there exists $\boldsymbol{x}^\star \in B^2$ such that $f(\boldsymbol{x}^\star) = 0$.*

The generalization, that we call k-Polygon Borsuk-Ulam Theorem, comes as a clean extension of Borsuk-Ulam where instead of assuming equivariance to a rotation of $180°$ degrees, we assume equivariance to a rotation of $(360/k)°$ degrees.

Informal Theorem 2 (k-Polygon Borsuk-Ulam Theorem). *Let $f : B^2 \to \mathbb{R}^2$ be a continuous function that is equivariant to a rotation of $\frac{360°}{k}$ degrees on the boundary. Then, there exists $\boldsymbol{x}^\star \in B^2$ such that $f(\boldsymbol{x}^\star) = 0$.*

Apart from its own mathematical interest, the k-Polygon Borsuk-Ulam Theorem is essential for our results, since it serves as a stepping stone towards showing all our topological PPA-p-completeness results. Also, it is the only one of our topological problems which is defined for any $k \geq 2$, and thus the only problem which we are able to relate to the classes PPA-k, for general k.

To prove the k-Polygon Borsuk-Ulam Theorem, we deviate from the topological techniques that have been used in the proof of similar extensions of the Borsuk-Ulam Theorem [Bárány et al., 1981] and we instead provide a combinatorial proof of a corresponding generalization of Tucker's lemma. We call this lemma k-Polygon Tucker's Lemma and an informal statement follows.

Informal Theorem 3 (k-Polygon Tucker's Lemma). *For $k \geq 3$, let T be a triangulation of B^2 with k-fold rotationally symmetric boundary. Suppose that every vertex $\mathbf{x} \in T$ has a color $\lambda(\mathbf{x}) \in \mathbb{Z}_k$ such that λ is equivariant to a rotation of $\frac{360°}{k}$ degrees on the boundary. Then, in T there exists (i) a trichromatic triangle, or (ii) an edge with distinct non-consecutive colors.*

The coloring function λ is equivariant to a rotation of $\alpha°$ degrees on the boundary if whenever the argument $\boldsymbol{x}$ is rotated by $\alpha°$, the color is increased by $1 \pmod k$. Arguably, for $k = 3$ the above statement of k-Polygon Tucker's Lemma bears more resemblance to Sperner's Lemma than to the original version of Tucker's Lemma, because the solution is necessarily a trichromatic triangle. However, this is due to the fact that we are considering only the two-dimensional case here. The connection with the original version of Tucker's Lemma will become more apparent when we present other high-dimensional modulo-p generalizations of Tucker's Lemma.

In the proof of k-Polygon Tucker's Lemma, the only inefficient step is the use of a *modulo-k* counting argument. A simple way to visualize this argument is to imagine that if a space has cardinality that is non-zero modulo k and if we can group points in this space that are non-solutions into groups of size k, then in this space there should exist a solution. This kind of existential argument has been formalized by Papadimitriou in his seminal paper [Papadimitriou, 1994] and there have been various instantiations of this principle from which we can define corresponding computational problems [Göös et al., 2020, Hollender, 2019]. In this paper, we mostly rely on the following instantiation defined by Hollender [2019]:

> Imbalance-mod-k : Given a directed graph and a vertex that is *imbalanced-mod-k*, i.e., $(\text{out-degree}) - (\text{in-degree}) \neq 0 \pmod k$, find another such vertex.

Our main technical contribution in this section is to show that the computational problem associated with k-Polygon Tucker's Lemma is polynomial time equivalent to the computational version of Imbalance-mod-k. Towards this goal, we provide a generalization of the standard *path-following* proof of Tucker's lemma by Freund and Todd [1981]. Importantly, in our proof (which is a reduction to Imbalance-mod-k) the edges of the path are *directed* by using a consistent direction of triangulations (inspired by the idea of Freund [1984]). This technique was in fact incorrectly applied in the past to the case of $k = 2$, leading to a false statement of PPAD-membership for Borsuk-Ulam. However, it turns out that the technique is very relevant in showing the equivalence between topological problems and modulo-k arguments for $k > 2$. We illustrate the appropriate way to use this technique via the Imbalance-mod-k problem and we believe that this will be useful for future reductions in PPA-k.

The computational equivalence between k-Polygon Tucker's Lemma and Imbalance-mod-k together with the computational equivalence of k-Polygon Tucker's Lemma and the k-Polygon Borsuk-Ulam implies the following theorem, which is our main result in this section.

Informal Theorem 4. *If k is a prime power, the computational problem associated with the k-Polygon Borsuk-Ulam Theorem is* PPA-k*-complete. If k is not a prime power, then it is complete for a subclass of* PPA-k*, denoted by* PPA-k[#1]*.*

The reason for this differentiation depending on whether k is a prime power or not, is that we actually show equivalence of k-Polygon Tucker's Lemma with a *special case* of Imbalance-mod-k. If k is a prime power, then this special case is in fact PPA-k-complete, but in general it can be weaker.

The formal definitions and the complete proofs, including the proof of Informal Theorem 4, about the k-Polygon Borsuk-Ulam Theorem appear in Section 3.

1.1.2 Complexity of Finding Solutions to the BSS Theorem

In this section, we present our results regarding the computational complexity of finding solutions guaranteed to exist by the BSS Theorem, a famous generalization of the celebrated Borsuk-Ulam Theorem. The BSS Theorem should be thought of as the corresponding k-Polygon Borsuk-Ulam in higher dimensions. We clarify though that the BSS Theorem only works with $k = p$ where p is a prime, and it applies only when the number of dimensions is a multiple of $p - 1$. Hence, for $p \geq 5$ the p-Polygon Borsuk-Ulam Theorem does not follow directly from the BSS Theorem. This is why we consider it a separate result and devote a separate section to it.

When moving to more than two dimensions, we need to find an equivariance notion corresponding to the equivariance of a rotation that we defined in the previous section. A fundamental feature of a rotation in the plane is that it is a *free action* on the boundary, i.e., there is no point on the boundary S^1 that remains fixed if we apply the rotation. This free action property of rotations is crucial in the proof of k-Polygon Borsuk-Ulam Theorem and without this property the theorem does not hold. Unfortunately, in higher dimensions the rotations with respect to any axis no longer possess this property, as they always have a fixed point on the boundary S^{m-1} of the m-dimensional ball B^m. Thus, in higher dimensions other operations acting freely on S^{m-1} emerge.

For the case of $k = 2$, there is a very simple generalization of the rotation by $180°$ that is a free action in any number of dimensions, namely, *the point reflection with respect to the origin*, i.e., $x \mapsto -x$. Hence, we have the following informal statement of the Borsuk-Ulam Theorem for a general number of dimensions.

Informal Theorem 5 (Borsuk-Ulam Theorem). *Let $f : B^m \to \mathbb{R}^m$ be a continuous function that on the boundary is equivariant to point reflection with respect to the origin. Then, there exists $\boldsymbol{x}^\star \in B^m$ such that $f(\boldsymbol{x}^\star) = 0$.*

Observe that the *order* of the point reflection operation is equal to $k = 2$, since if we apply the same operation twice, we return to the same point. It is also trivial to see that the rotation by $360°/k$ degrees has order k, since after k times of applying this operation we return to the same point and if we apply this operation less than k times, then we end up on a different point. These observations together with the requirement for a free action suggest that in order to generalize the k-Polygon Borsuk-Ulam theorem to higher dimensions we need a free action of order k on the boundary S^{m-1} of the m-dimensional ball B^m. Unfortunately, finding such operations is not as easy as in the case $k = 2$. In particular, for $m = 2\ell + 1$ the sphere $S^{2\ell}$ has a free action of order k only for $k = 2$ [Hatcher, 2002]. The starting point of the BSS Theorem is defining an operation α_p that has order p, where p is a prime number, and defines a free action on the sphere $S^{n(p-1)-1}$. These restrictions on α_p are the reason why, as we mentioned in the beginning of the section, BSS only applies to dimensions that are multiples of $p - 1$ and for $k = p$ where p is a prime number. Using the operation α_p, we can informally state the BSS Theorem as follows:

Informal Theorem 6 (BSS Theorem [Bárány, Shlosman, and Szücs, 1981]). *Let $f : B^{n(p-1)} \to \mathbb{R}^{n(p-1)}$ be a continuous function that is equivariant with respect to α_p on the boundary. Then, there exists $\boldsymbol{x}^\star \in B^{n(p-1)}$ such that $f(\boldsymbol{x}^\star) = 0$.*

The original proof of the BSS Theorem [Bárány et al., 1981] goes through the definition of indices of functions in algebraic topology. One of our main contributions is to provide a combinatorial proof of the BSS Theorem. As in the case of k-Polygon Borsuk-Ulam, we provide a combinatorial proof of the BSS Theorem via the corresponding version of Tucker's Lemma, which we call BSS Tucker's Lemma and we informally state below.

Informal Theorem 7 (BSS Tucker's Lemma). *Let p be a prime and T be a triangulation of $B^{n(p-1)}$ with an α_p-symmetric boundary. Suppose that every vertex $\boldsymbol{x} \in T$ has a color $\lambda(\boldsymbol{x}) \in \mathbb{Z}_p \times [n]$ such that λ is equivariant with respect to α_p on the boundary. Then, there exists a $(p-1)$-simplex in T that has all the colors $(1, j), \dots (p, j)$ for some $j \in [n]$.*

Our main technical contribution in this section is to show the following statements.

- ▷ The computational problem that is associated with BSS Tucker's Lemma is polynomial time equivalent to the computational problem associated with the BSS Theorem (Theorem 4.8).
- ▷ The computational problem associated with p-Polygon Tucker's Lemma is reducible to the computational problem associated with BSS Tucker's Lemma (Theorem 4.10).
- ▷ The computational problem associated with BSS Tucker's Lemma is reducible to the computational problem associated with $\mathbb{Z}_p$-star Tucker's Lemma (Proposition 5.4).

$\mathbb{Z}_p$-star Tucker's Lemma is an existence theorem that we define informally in the next section and is the basic building block for proving the membership of Necklace Splitting in PPA-p. As we will see in the next section, we prove that the computational problem associated with $\mathbb{Z}_p$-star Tucker's Lemma is inside PPA-p. This result combined with Informal Theorem 4, which shows the PPA-p-completeness of the p-Polygon Borsuk-Ulam Theorem, and the equivalence of the p-Polygon Borsuk-Ulam Theorem and p-Polygon Tucker's lemma, implies the main result of this section.

Informal Theorem 8. *The computational problem associated with the BSS Theorem is* PPA-p*-complete.*

We defer the formal definition of the computational problem associated with the BSS Theorem and the complete proofs, including the proof of Informal Theorem 8, about the BSS Theorem appear in Section 4.

1.1.3 Necklace Splitting with p thieves is in PPA-p

Our main goal in the last part of the paper is to show that the computational problems associated with the p-Necklace Splitting Theorem and the Consensus-$1/p$-Division Theorem are in PPA-p. Towards this goal we provide a full combinatorial proof for both of these problems that simplifies the only existing combinatorial proof by Meunier [2014].

Our proof uses a different generalization of Tucker's Lemma which we call $\mathbb{Z}_p$-star Tucker's Lemma. Both the statement of $\mathbb{Z}_p$-star Tucker Lemma and the proof are simple enough so that they do not invoke the involved definition of *simplotopal* complexes of Meunier [2014]. These simplotopal complexes were used by Meunier in order to obtain a direct proof of the Necklace-Splitting theorem, without proving its continuous version. In contrast, we use standard simplicial complexes, since we are not only interested in the Necklace Splitting theorem, but also in its continuous generalization: the Consensus-$1/k$-Division Theorem (informally defined below).

Recall that in the k-Necklace Splitting problem, k thieves are aiming to split an open necklace containing n beads of t different colors, with exactly $k \cdot a_i$ beads of color i, for some $a_i \in \mathbb{N}$, such that each thief receives exactly a_i beads of color i. The k-Necklace Splitting Theorem states the following.

Informal Theorem 9 (k-Necklace Splitting Theorem [Alon, 1987])**.** *The k-Necklace Splitting problem always has a solution with $(k-1)t$ cuts.*

The Consensus-$1/k$-Division problem resembles the continuous version of the k-Necklace Splitting problem. In this problem, each one of t agents has a probability measure μ_i over the unit interval $[0, 1]$. The goal is to cut the interval $[0, 1]$ into pieces and assign one of k possible colors to each piece such that every agent measures the total mass of each different color the same.

Informal Theorem 10 (Consensus-$1/k$-Division Theorem [Alon, 1987, Simmons and Su, 2003])**.** *The Consensus-$1/k$-Division problem always has a solution with $(k-1)t$ cuts.*

Both the k-Necklace Splitting Theorem and the Consensus-$1/k$-Division Theorem have significant applications to combinatorics and social choice; see Section 1.2.

As we have already mentioned, the proof that for prime p the computational problems associated with the p-Necklace Splitting Theorem and the Consensus-$1/p$-Division Theorem are inside PPA-p, is based on a different generalization of Tucker's Lemma for modulo-p arguments, $\mathbb{Z}_p$-star Tucker's Lemma. The main difference of $\mathbb{Z}_p$-star Tucker's Lemma with BSS Tucker's Lemma is the domain on which we define the triangulation. In the case of BSS Tucker's Lemma, the triangulation is defined over a convex domain and hence is homeomorphic with the ball B^m. On the other hand, the triangulation for $\mathbb{Z}_p$-star Tucker's Lemma is defined over a *star-convex* set which is not homeomorphic to the ball B^m anymore. This d-dimensional domain, which we denote by R_p^d, is a slightly modified version of the domain used by Meunier [2014] in his combinatorial proof of the p-Necklace Splitting Theorem. It admits a natural free action θ_p of order p. We can informally state $\mathbb{Z}_p$-star Tucker's Lemma similarly to BSS Tucker's lemma as follows.

Informal Theorem 11 ($\mathbb{Z}_p$-STAR TUCKER'S LEMMA)**.** *Let p be a prime and T be a triangulation of $R_p^{t(p-1)}$. Suppose that every vertex $\boldsymbol{x} \in T$ has a color $\lambda(\boldsymbol{x}) \in \mathbb{Z}_p \times [t]$ such that λ is equivariant with respect to θ_p on the boundary. Then, there exists a $(p-1)$-simplex of T that has all the colors $(1, j), \ldots (p, j)$ for some $j \in [t]$.*

Our main technical contributions in this section are summarized in the following statements.

- ▷ The computational problems that are associated with the p-Necklace Splitting Theorem and the Consensus-$1/p$-Division Theorem are polynomial time reducible to $\mathbb{Z}_p$-star Tucker's Lemma (Theorem 6.2).
- ▷ The computational problem associated with $\mathbb{Z}_p$-star Tucker's Lemma is inside PPA-p (Theorem 5.3). This is again proved by a reduction to IMBALANCE-MOD-p, but this time we construct a *weighted* directed graph.
- ▷ The computational problem associated with BSS Tucker's Lemma is polynomial time reducible to the computational problem associated with $\mathbb{Z}_p$-star Tucker's Lemma (Proposition 5.4).

The above statements combined with the results in the previous sections imply our main result for this part of the paper.

Informal Theorem 12. *The computational problem associated with $\mathbb{Z}_p$-star Tucker's Lemma is* PPA-p*-complete. The computational problems that are associated with the p-Necklace Splitting Theorem and the Consensus-*$1/p$*-Division Theorem are in* PPA-p*.*

As a corollary of this result, we also obtain that for general $k \geq 2$, k-Necklace Splitting and Consensus-$1/k$-Division lie in PPA-k *under Turing reductions*. In particular, if $k = p^r$ is a prime power, then the corresponding problems lie in PPA-p. Furthermore, our reductions also provide a new combinatorial proof of the Necklace Splitting theorem, that is conceptually simpler and does not use any involved machinery.

The formal definitions and the complete proofs, including the proof of Informal Theorem 12, about $\mathbb{Z}_p$-star Tucker's Lemma, the p-Necklace Splitting Theorem and the Consensus-$1/p$-Division Theorem appear in Section 5 and Section 6.

Our results are summarized in Table 1, where we also highlight where they fit in the computational landscape of the classes of interest. Relevant further related work is discussed in the next section.

1.2 Discussion and Further Related Work

Computational Classes: As mentioned earlier, among the classes of TFNP, PPAD has been the most successful in capturing the complexity of well-known problems. Besides the complexity of computing a Nash equilibrium [Daskalakis et al., 2009, Chen et al., 2009b, Rubinstein, 2018], other PPAD-complete problems are computing equilibria in markets [Chen et al., 2009a, 2013], versions of envy-free cake cutting [Deng et al., 2012] and fixed-point theorems [Mehta, 2014, Goldberg and Hollender, 2019].

For PPA, the recent results by Filos-Ratsikas and Goldberg [2018, 2019] have solidified the status of the class as one that contains natural problems. In particular, they showed that 2-thief Necklace Splitting is PPA-complete; the proof goes via its continuous version, the *Consensus-Halving* problem of Simmons and Su [2003][3] Our PPA-p-membership result for the problem with

[3]The hardness result of Filos-Ratsikas and Goldberg [2018, 2019] for the Consensus-Halving problem was strengthened recently by Filos-Ratsikas et al. [2020] to the case of simpler measures.

	PPAD	PPA	PPA-p ($p \geq 3$)
Topological Existence Theorem	BROUWER [Papadimitriou, 1994] [Chen and Deng, 2009] HAIRY-BALL [Goldberg and Hollender, 2019]	BORSUK-ULAM [Papadimitriou, 1994] [Aisenberg et al., 2020]	p-POLYGON-BORSUK-ULAM p-BSS [This Work]
Combinatorial Lemma	SPERNER [Papadimitriou, 1994] [Chen and Deng, 2009]	TUCKER [Papadimitriou, 1994] [Aisenberg et al., 2020]	p-POLYGON-TUCKER p-BSS-TUCKER $\mathbb{Z}_p$-STAR-TUCKER [This Work]
Notable Problems	NASH [Daskalakis et al., 2009] [Chen et al., 2009b] MARKET-EQUILIBRIUM [Chen et al., 2009a, 2013] *and many more...*	2-NECKLACE-SPLITTING CONSENSUS-HALVING DISCRETE-HAM-SANDWICH [Filos-Ratsikas and Goldberg, 2019]	SYMMETRIC-CHEVALLEY-MOD-p [Göös et al., 2020] p-NECKLACE-SPLITTING CONSENSUS-$1/p$-DIVISION [Membership: This Work] [Hardness: Open]

Table 1: An overview of the computational landscape for the related TFNP classes.

p thieves also uses the continuous variant, termed as *Consensus-*$1/p$*-Division* in [Simmons and Su, 2003]; we note that Alon [1987] used the same problem in his existence proof, referring to it as a *generalized Hobby-Rice Theorem*. As we explained earlier, Filos-Ratsikas and Goldberg [2019] conjectured that the problem with p thieves is complete for PPA-p.

The classes PPA-p were introduced by Papadimitriou [1994] for any prime p, in the context of classifying a computational version of the Chevalley-Warning theorem [Chevalley, 1935, Warning, 1935]. He proved that the corresponding problem CHEVALLEY-MOD-p lies in PPA-p. Recently, Göös et al. [2020] showed that an explicit version of the problem is complete for PPA-p, therefore obtaining the first PPA-p-completeness result for a natural problem. The authors of [Göös et al., 2020], as well as Hollender [2019], independently also extended the definition of the classes PPA-k to any $k \geq 2$, and provided several characterizations in terms of their defined-for-primes counterparts. Hollender [2019] also investigated the connection with the classes PMOD-k, which bear strong resemblance to PPA-k, and were defined seemingly independently of Papadimitriou's work by Johnson [2011]. From the work of Hollender [2019], we primarily make use of the PPA-k-complete computational problem IMBALANCE-MOD-k, a generalization of the PPAD-complete problem IMBALANCE [Beame et al., 1998, Goldberg and Hollender, 2019].

Necklace Splitting: The origins of the Necklace Splitting problem (Informal Theorem 9), and its continuous variant (Informal Theorem 10), can be traced back to work of Neyman [1946] and Hobby and Rice [1965]. The problem was firstly phrased as a necklace splitting problem by Bhatt and Leiserson [1982]. Later on, Goldberg and West [1985] and Alon and West [1986] proved the first existence results for two thieves, and Alon [1987] extended the result to the case of any number $k \geq 2$ of thieves. For two thieves, the continuous version was studied independently by Simmons and Su [2003], who termed the problem as "Consensus-Halving" and came up with a combinatorial, constructive proof of existence; this proof was later modified into a PPA membership proof by Filos-Ratsikas et al. [2018]. The importance of obtaining combinatorial proofs of existence was highlighted in a series of papers [de Longueville and Živaljević, 2006, Meunier, 2008, 2014, Meunier and Neveu, 2012]; note that the proof of Alon [1987] uses two results of Bárány et al. [1981] and is therefore not combinatorial. The first fully combinatorial proof for Necklace Splitting (according to the definition of Ziegler [2002]) is due to Meunier [2014]. However, the proof comes up with a rather involved construction based on the notion of simplotopal complexes, and thus is notably hard to follow without an advanced understanding of the theory of simplicial complexes. Since our proof is based on a reduction, it is also inherently combinatorial. Very recently, Alon and Graur [2020] studied approximate versions of both the continuous and the discrete variants, and provided polynomial-time algorithms for either large enough approximations or a larger number of cuts (i.e., cases not captured by the hardness results of [Filos-Ratsikas and Goldberg, 2018, 2019, Filos-Ratsikas et al., 2020]).

The BSS Theorem: The BSS Theorem (Informal Theorem 6, Theorem 4.2), due to [Bárány et al., 1981], is perhaps the most well-known generalization of the Borsuk-Ulam theorem. Besides [Alon, 1987], it has been used to prove existence of other interesting problems, including a generalization of the Kneser-Lovász Theorem [Kneser, 1955, Lovász, 1978] due to Alon et al. [1986], a generalized van Kampen-Flores Theorem [Sarkaria, 1991] and the generalization to Tverberg's Theorem, proven in [Bárány et al., 1981]. We believe that our PPA-p-completeness result paves the way for studying the complexity of those problems as well.

2 Preliminaries

2.1 Total NP Search Problems

Let $\{0,1\}^*$ denote the set of all finite length bit-strings. For $x \in \{0,1\}^*$, let $|x|$ be its length. A computational search problem is given by a binary relation $R \subseteq \{0,1\}^* \times \{0,1\}^*$. The associated problem is: given an instance $x \in \{0,1\}^*$, find a $y \in \{0,1\}^*$ such that $(x,y) \in R$, or return that no such y exists. The search problem R is in FNP (*Functions in NP*), if R is polynomial-time computable (i.e., $(x,y) \in R$ can be decided in polynomial time in $|x| + |y|$) and there exists some polynomial p such that $(x,y) \in R \implies |y| \leq p(|x|)$. Thus, FNP is the search problem version of NP.

The class TFNP (*Total Functions in NP* [Megiddo and Papadimitriou, 1991]) contains all FNP search problems R that are *total*: for every $x \in \{0,1\}^*$ there exists $y \in \{0,1\}^*$ such that $(x,y) \in R$. Note that the totality of problems in TFNP does not rely on any "promise". Instead, there is a *syntactic* guarantee of totality: for any instance in $\{0,1\}^*$, there is always at least one solution.

Let R and S be total search problems in TFNP. We say that R (many-one) reduces to S, denoted $R \leq S$, if there exist polynomial-time computable functions f, g such that

$$(f(x), y) \in S \implies (x, g(x,y)) \in R.$$

Note that if S is polynomial-time solvable, then so is R. We say that two problems R and S are (polynomial-time) equivalent, if $R \leq S$ and $S \leq R$.

Sometimes a more general notion of reduction is used. A Turing reduction from R to S is a polynomial-time oracle Turing machine that solves problem R with the help of queries to an oracle for S. Note that a Turing reduction that only makes a single oracle query immediately yields a many-one reduction.

2.2 The Classes PPA-k

For $k \geq 2$, PPA-k is a subclass of TFNP that aims to capture the complexity of TFNP problems whose totality is proved by using an argument modulo k. The classes PPA-p (for prime p) were introduced by Papadimitriou [1994]. The case $p = 2$ corresponds to *parity arguments*, and in that case the class PPA-2 is simply called PPA. Recently, the definition of PPA-k was extended to any $k \geq 2$ by Göös et al. [2020], Hollender [2019]. The class PPA-k is defined as the class of TFNP problems that reduce to the following problem.

Definition 1 (Bipartite-mod-k [Papadimitriou, 1994])**.** Let $k \geq 2$. The problem Bipartite-mod-k is defined as: given a Boolean circuit $C : \{0,1\} \times \{0,1\}^n \to (\{0,1\} \times \{0,1\}^n)^k$ that computes a bipartite graph on the vertex set $(\{0\} \times \{0,1\}^n, \{1\} \times \{0,1\}^n)$ with $|C(00^n)| \in \{1,\ldots,k-1\}$, find

- $x \neq 00^n$ such that $|C(x)| \notin \{0,k\}$
- or x, y such that $y \in C(x)$ but $x \notin C(y)$.

The circuit C computes a bipartite graph as follows. The output of circuit C on some input x is a list of k bit-strings, each of length $n+1$. If $x \in \{0\} \times \{0,1\}^n$, then we let $C(x)$ denote the set of distinct bit-strings that appear in that list and that lie in $\{1\} \times \{0,1\}^n$. Similarly, if $x \in \{1\} \times \{0,1\}^n$, then $C(x)$ denotes the set of distinct bit-strings that appear in that list and that lie in $\{0\} \times \{0,1\}^n$. For any $x, y \in \{0,1\} \times \{0,1\}^n$, there exists an edge between x and y if and only if $y \in C(x)$ and $x \in C(y)$. It is easy to see that this indeed defines a bipartite graph.

Definition 2. Let $k \geq 2$ and $1 \leq \ell \leq k-1$. The problem Bipartite-mod-$k[\#\ell]$ is defined as Bipartite-mod-k (Definition 1) but with the additional condition $|C(00^n)| = \ell$.

Note that this condition can be enforced syntactically and so this problem also lies in TFNP (see [Papadimitriou, 1994] for a definition of "syntactic").

Definition 3 (PPA-$k[\#\ell]$)**.** Let $k \geq 2$ and $1 \leq \ell \leq k-1$. The class PPA-$k[\#\ell]$ is defined as the set of all TFNP problems that many-one reduce to Bipartite-mod-$k[\#\ell]$.

The following result relates these special subclasses to the main ones.

Proposition 2.1 ([Hollender, 2019])**.** PPA-$k[\#1] = \cap_{p \in PF(k)}$PPA-$p$*, where $PF(k)$ denotes the set of prime factors of k.*

In this paper, we will also use the following definition of PPA-k, which was shown to be equivalent to the standard one [Hollender, 2019]. The class PPA-k is the set of all TFNP problems that reduce to the following problem.

Definition 4. Let $k \geq 2$. The problem Imbalance-mod-k is defined as: given Boolean circuits $S, P : \{0,1\}^n \to (\{0,1\}^n)^k$ with $|S(0^n)| - |P(0^n)| \neq 0 \mod k$, find

- $x \neq 0^n$ such that $|S(x)| - |P(x)| \neq 0 \mod k$
- or x, y such that $y \in S(x)$ but $x \notin P(y)$, or $y \in P(x)$ but $x \notin S(y)$.

For $1 \leq \ell \leq k-1$, Imbalance-mod-$k[\#\ell]$ is defined with the additional condition $|S(0)| - |P(0)| = \ell$.

We obtain a seemingly more general version of this problem by allowing the edges to have integer weights. In that case the imbalance of a vertex is measured as the difference of the weights of all incoming edges and the weights of all outgoing edges. It is easy to see that this problem is in fact equivalent to Imbalance-mod-k. First, all the weights can be assumed to be in $\{0, 1, \ldots, k-1\}$, since we can reduce them modulo k. Next, we can split an edge with weight ℓ into ℓ copies of the edge. Finally, to ensure that we don't have multi-edges, we add a new vertex in the middle of every edge. Note that the new vertex will be balanced by construction, and will thus not introduce any new solutions.

It is easy to see that in all the aforementioned problems a solution is always guaranteed to exist. The search problems are not trivial though, because the graph can have exponential size with respect to its description. Indeed, the graph is given by Boolean circuits that compute the successors and predecessors of every vertex.

We make use of the following known properties of these classes:

Proposition 2.2 (Göös et al. [2020], Hollender [2019]). *It holds that:*

- *for any prime p and any $r \geq 1$,* PPA-p^r = PPA-p
- *for any $k, \ell \geq 1$,* PPA-$k \subseteq$ PPA-$k\ell$

Topological Definitions and Details: Our results in the next sections will require several definitions from topology, as well as the corresponding notation. The more experienced reader might already be familiar with some of these concepts, but all the relevant details are included in Appendix A. There, we also define our *value* and *index* functions, which allow us to enumerate over simplices, and access simplices that contain a point $\mathbf{x}$ in the domain respectively.

3 k-Polygon Borsuk-Ulam: a PPA-k[#1]-complete Problem in 2D-space

In this section we present a generalization of the Borsuk-Ulam Theorem in two dimensional space. Surprisingly this theorem is not captured by the BSS Theorem and hence has its own topological interest. Our proof of this theorem is combinatorial and is based on a generalization of Tucker's Lemma which we call *Polygon Tucker's Lemma*, hence it is very different from the proof of the BSS Theorem. Our main result is that k-Polygon Borsuk-Ulam is complete for PPA-k[#1]. This gives the first topological characterization of the classes PPA-k[#1] but also reveals in a very intuitive way the relation between the different PPA-k[#1] classes and PPAD. Recall that when $k = p^r$ is a prime power, PPA-k[#1] = PPA-k = PPA-p.

We start this section with a unified description of Brouwer's Fixed Point Theorem, the Borsuk-Ulam Theorem and our generalization: the k-Polygon Borsuk-Ulam Theorem. For this we will need the following definition.

Definition 5 (Rotation Operator). We define the k-th *rotation operator* $\theta_k : \mathbb{R}^2 \to \mathbb{R}^2$ as follows: $\theta_k(\boldsymbol{x}) = R_k \boldsymbol{x}$, where R_k is the following two dimensional rotation matrix

$$R_k = \begin{bmatrix} \cos\left(-\frac{2\pi}{k}\right) & -\sin\left(-\frac{2\pi}{k}\right) \\ \sin\left(-\frac{2\pi}{k}\right) & \cos\left(-\frac{2\pi}{k}\right) \end{bmatrix}.$$

In other words, θ_k corresponds to a clockwise rotation by an angle of $2\pi/k$.

We continue with a statement of Brouwer's Fixed Point Theorem that is different from the standard statement, but is well known to be equivalent to that (e.g., see [Matoušek, 2008]).

Theorem 3.1 (Brouwer's Fixed Point Theorem). *Let $f : B^2 \to \mathbb{R}^2$ be a continuous function such that $f(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \partial B^2$. Then there exists $\boldsymbol{x}^\star \in B^2$ such that $f(\boldsymbol{x}^\star) = 0$.*

Next, using the same language, we give a statement of the Borsuk-Ulam Theorem.

Theorem 3.2 (Borsuk-Ulam Theorem). *Let $f : B^2 \to \mathbb{R}^2$ be a continuous function such that $f(\theta_2(\boldsymbol{x})) = \theta_2(f(\boldsymbol{x}))$ for all $\boldsymbol{x} \in \partial B^2$. Then there exists $\boldsymbol{x}^\star \in B^2$ such that $f(\boldsymbol{x}^\star) = 0$.*

It is clear from the above expression that the Borsuk-Ulam Theorem is a generalization of Brouwer's Fixed Point Theorem. This observation is in line with the fact that PPAD $\subseteq$ PPA, since Brouwer is complete for PPAD and Borsuk-Ulam is complete for PPA. We now present our extension, that we call k-Polygon Borsuk-Ulam Theorem.

Theorem 3.3 (k-Polygon Borsuk-Ulam Theorem). *Let $f : B^2 \to \mathbb{R}^2$ be a continuous function such that $f(\theta_k(\boldsymbol{x})) = \theta_k(f(\boldsymbol{x}))$ for all $\boldsymbol{x} \in \partial B^2$. Then there exists $\boldsymbol{x}^\star \in B^2$ such that $f(\boldsymbol{x}^\star) = 0$.*

As we will see the k-Polygon Borsuk-Ulam Theorem is also a generalization of Brouwer's Fixed Point Theorem and it is complete for PPA-k[#1] which is also in line with the fact that PPAD $\subseteq$ PPA-k[#1]. Another interesting fact about the k-Polygon Borsuk-Ulam Theorem is that it does not directly follow from the traditional generalization of the Borsuk-Ulam Theorem, namely the BSS Theorem, as we will see in the next section.

3.1 k-Polygon Tucker's Lemma and k-Polygon Borsuk-Ulam in PPA-k[#1]

In this section, we define k-Polygon Tucker's Lemma and we prove that it is equivalent to the k-Polygon Borsuk-Ulam Theorem. Additionally, we provide a combinatorial proof of k-Polygon Tucker's Lemma using a modulo-k argument. This combinatorial proof puts both k-Polygon Tucker and k-Polygon Borsuk-Ulam in the class PPA-k[#1].

Before showing the equivalence of the two statements, we provide some necessary notation and definitions.

Definition 6 (k-Polygon & Nice Triangulation). For $k \geq 3$, let W_k be the regular k-polygon, i.e., regular k-gon, centered at $\mathbf{0} \in \mathbb{R}^2$ with radius 1. Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ denote the vertices of W_k, ordered such that $\theta_k(\mathbf{u}_i) = \mathbf{u}_{i+1 \pmod k}$ for all $i \in [k]$. We define $T^\star$ to be the triangulation of W_k that includes the simplices $\sigma_i = \mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i, \boldsymbol{u}_{i+1 \pmod k}\})$ for $i \in [k]$. We call a triangulation T *nice* if it satisfies the following two properties:

- it is a refinement of $T^\star$, and
- it is symmetric with respect to θ_k on the boundary. This means that for every edge $\psi \in T$ such that $\psi \subseteq \partial W_k$ it holds that $\theta_k(\psi) \in T$.

Theorem 3.4 (k-Polygon Tucker's Lemma). *For $k \geq 3$, let W_k be a k-regular polygon. Fix some nice triangulation T of W_k. Suppose that every vertex $\mathbf{x} \in T$ has a label $\lambda(\mathbf{x}) \in \mathbb{Z}_k$ such that for any $\mathbf{y} \in \partial T$ we have $\lambda(\theta_k(\mathbf{y})) = \lambda(\mathbf{y}) + 1 \pmod k$. Then at least one of the following exists: (1) a simplex $\sigma \in T$ with vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ such that all the labels $\lambda(\boldsymbol{v}_1)$, $\lambda(\boldsymbol{v}_2)$, $\lambda(\boldsymbol{v}_3)$ are different, or (2) an edge $\psi \in T$ with vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ such that $\lambda(\boldsymbol{v}_1) - \lambda(\boldsymbol{v}_2) \pmod k \notin \{0, 1, -1\}$.*

Remark 1. The above theorem can be proved if we invoke Dold's Theorem from algebraic topology Dold [1983]. However this proof is not a constructive proof and hence it cannot be used for our purposes and for this reason we reprove this theorem using a constructive combinatorial proof. Of course, this is only a special case of the general Dold's Theorem and it is an interesting open problem what is the relation of Dold's Theorem with the subclasses of TFNP as we state in the Conclusions section.

3.1.1 Equivalence of k-Polygon Tucker and k-Polygon Borsuk-Ulam

We start by showing that k-Polygon Tucker's Lemma is implied by k-Polygon Borsuk-Ulam Theorem and then we also show the converse, in Lemma 3.5 and Lemma 3.6.

Lemma 3.5. *k-Polygon Borsuk-Ulam (Theorem 3.3) implies k-Polygon Tucker's Lemma (Theorem 3.4).*

Proof. We interpret each label $i \in \mathbb{Z}_k$ as the vector $\mathbf{u}_i$, which is the i-th vertex of the polygon W_k. Let $h : W_k \to W_k$ be the piecewise linear extension of the function that has value $\mathbf{u}_{\lambda(\boldsymbol{x})}$ on any vertex $\boldsymbol{x} \in T$. Finally we define $g : B^2 \to \mathbb{R}^2$ as the composition of h and a homeomorphism between W_k and B^2 that maps $\mathbf{0}$ to $\mathbf{0}$ and is θ_k-equivariant. Notice that by the definition of h and g and the equivariance assumption on λ, it holds that $g(\theta_k(\mathbf{x})) = \theta_k(g(\mathbf{x}))$ for $\mathbf{x} \in \partial B^2$. Then, it follows from Theorem 3.3 that there exists an $\mathbf{y}^\star \in B^2$ such that $g(\mathbf{y}^\star) = 0$ and hence there exists a $\mathbf{x}^\star \in W_k$ such that $h(\mathbf{x}^\star) = 0$.

Now, let $\sigma^\star$ be a full-dimensional simplex of T that contains $\mathbf{x}^\star$. If the vertices of $\sigma^\star$ have only two consecutive labels, say 1 and 2 (corresponding to vectors $\mathbf{u}_1$ and $\mathbf{u}_2$), then it is impossible to have $h(\mathbf{x}^\star) = 0$ since it is a non-zero linear interpolation of $\mathbf{u}_1$ and $\mathbf{u}_2$ and these vectors are linearly independent. Hence, it has to be that the vertices of $\sigma^\star$ have either two non-consecutive labels or three different labels and k-Polygon Tucker's Lemma follows. □

Lemma 3.6. *k-Polygon Tucker's Lemma (Theorem 3.4) implies k-Polygon Borsuk-Ulam (Theorem 3.3).*

Proof. Let $h : W_k \to \mathbb{R}^2$ be the function obtained from f by using a homeomorphism between W_k and B^2 that fixes $\mathbf{0}$ and is θ_k-equivariant. Using standard arguments, that are used in the proof of both Brouwer's Fixed Point Theorem via Sperner's Lemma and the proof of Borsuk-Ulam via Tucker's Lemma, it is enough if for every $\varepsilon > 0$ we find a point $\mathbf{x}$ such that $\|h(\mathbf{x})\| \leq \varepsilon$, for details we refer to [Matoušek, 2008]. Since h is a continuous function in a compact set, it is also uniformly continuous. Thus for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $\mathbf{x}, \mathbf{x}' \in W_k$, if $\|\mathbf{x} - \mathbf{x}'\|_2 < \delta$, then $\|h(\mathbf{x}) - h(\mathbf{x}')\| < \varepsilon/k$. Assume that T is a nice triangulation of W_k such that any simplex $\sigma \in T$ has diameter at most δ.

We define the labeling $\lambda(\mathbf{x}) = i$ to be equal to the index of the vertex $\boldsymbol{u}_i$ of W_k that is closest to $h(\mathbf{x})$. We break ties between i and $i+1$, by picking i. Note that the only other kind of tie that can occur is if $h(\mathbf{x}) = 0$. In that case all labels are tied. If $\mathbf{x} = 0$, we pick one arbitrarily. Otherwise, i.e., if $\mathbf{x} \neq 0$, we apply the same rule as for $h(\mathbf{x})$ above, but for $\mathbf{x}$ instead. It is easy to check that this tie-breaking is $\mathbb{Z}_k$-equivariant. For any $\mathbf{x} \in \partial W_k$ because of the equivariance assumption on g we have that $\lambda(\theta_k(\mathbf{x})) = \lambda(\mathbf{x}) + 1 \pmod k$ and hence the assumptions of k-Polygon Tucker's Lemma are satisfied. Now we distinguish two cases.

$\boldsymbol{k = 3}$. In this case, 3-Polygon Tucker's Lemma implies that there exists a simplex $\sigma \in T$ whose vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ contain all the labels 1, 2, and 3 respectively. This implies the following set of inequalities by the definition of the labeling λ

$$\|h(\boldsymbol{v}_1) - \boldsymbol{u}_1\| \leq \min(\|h(\boldsymbol{v}_1) - \boldsymbol{u}_2\|, \|h(\boldsymbol{v}_1) - \boldsymbol{u}_3\|)$$

$$\|h(\boldsymbol{v}_2) - \boldsymbol{u}_2\| \leq \min(\|h(\boldsymbol{v}_2) - \boldsymbol{u}_1\|, \|h(\boldsymbol{v}_2) - \boldsymbol{u}_3\|)$$

$$\|h(v_3) - u_3\| \le \min(\|h(v_3) - u_1\|, \|h(v_3) - u_2\|).$$

Hence, it is easy to see that the maximum angle between any of $h(v_1)$, $h(v_2)$ and $h(v_3)$ is at least $2\pi/3$. Now, assume for the sake of contradiction that for all v_1, v_2, v_3 it holds that

$$\|h(v_1)\| \ge \varepsilon,\ \|h(v_2)\| \ge \varepsilon,\ \|h(v_3)\| \ge \varepsilon.$$

This together with the fact that the maximum angle is at least $2\pi/3$ implies that the maximum distance is at least $2\sin(2\pi/6)\varepsilon$. But this implies that

$$\max(\|h(v_1) - h(v_2)\|, \|h(v_2) - h(v_3)\|, \|h(v_1) - h(v_3)\|) \ge \sqrt{3} \cdot \varepsilon > \frac{\varepsilon}{3}$$

which contradicts the definition of the triangulation T, where the vertices of the same simplex are at most δ far from each other and hence their images are at most $\varepsilon/k = \varepsilon/3$ far from each other. This implies that our assumption was wrong and hence for at least one $i \in [3]$ it holds that $\|h(v_i)\| \le \varepsilon$ and the result for this case follows.

$k > 3$. In this case, k-Polygon Tucker's Lemma implies that there exists an edge $\psi \in T$ whose vertices v_1 and v_2 have labels that differ by more that one, without loss of generality assume that these labels are 1 and 3 respectively. By the definition of the labeling λ this implies that

$$\|h(v_1) - u_1\| \le \min_i(\|h(v_1) - u_i\|), \quad \|h(v_2) - u_3\| \le \min_i(\|h(v_2) - u_i\|)$$

Hence, it is easy to see that the angle between $h(v_1)$ and $h(v_2)$ is at least $2\pi/k$. Now, for sake of contradiction we assume that

$$\|h(v_1)\| \ge \varepsilon,\ \|h(v_2)\| \ge \varepsilon.$$

This together with the fact that the angle is at least $2\pi/k$ implies that the distance $\|h(v_1) - h(v_2)\|$ is at least $2\sin(2\pi/k)\varepsilon > \varepsilon/k$ which contradicts the definition of T.

Therefore, in both cases for every $\varepsilon > 0$ we can find a $v \in W_k$ such that $\|h(v)\| \le \varepsilon$. This, by standard arguments and the compactness of W_k, implies that there exists a point $x^\star \in W_k$ such that $h(x^\star) = \mathbf{0}$ and hence the result follows. □

3.1.2 Proof of k-Polygon Tucker's Lemma

In this section, we prove k-Polygon Tucker's Lemma. A corollary of our proof combined with Lemma 3.6 is that the computational problems associated with k-Polygon Tucker's Lemma and k-Polygon Borsuk-Ulam both belong to PPA-k[#1]. The proof technique that we introduce here is a generalization of the combinatorial proof of Tucker's lemma given by Freund and Todd [1981].

We use a modulo-k argument to prove this theorem. We define a directed graph where the vertices correspond to the simplices of T, and we also identify the symmetric edges of T on the boundary of W_k as the same vertex. Then, in this graph we describe a rule for defining edges such that there exist three types of vertices:

1. the vertex that corresponds to the 0-dimensional simplex $\{\mathbf{0}\}$ which will have degree 1,
2. vertices that are balanced $\pmod k$, i.e., $(\text{out-degree}) - (\text{in-degree}) = 0 \pmod k$,
3. vertices that are different from $\{\mathbf{0}\}$ and are not balanced $\pmod k$.

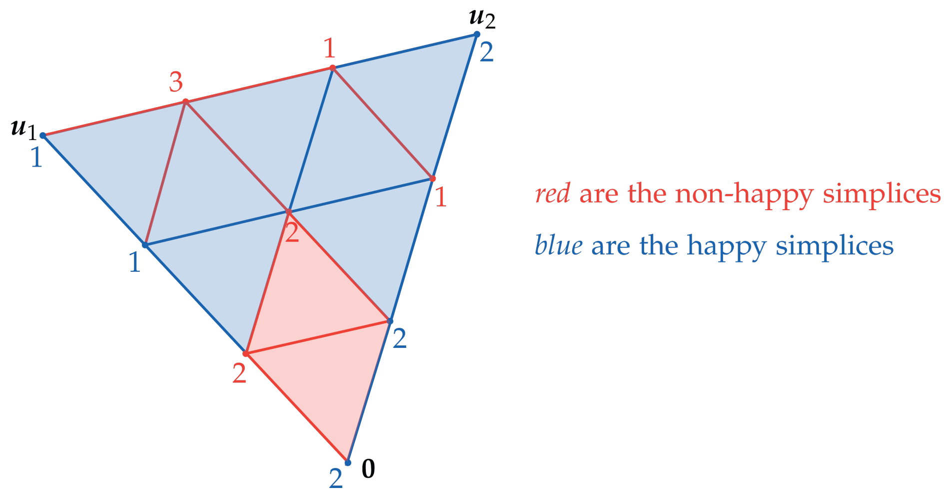

Figure 1: Example that shows happy and non-happy simplices in the cone $\mathsf{conv}(\{\boldsymbol{u}_1, \mathbf{0}, \boldsymbol{u}_2\})$.

Due to a simple modulo-k argument and because $\{\mathbf{0}\}$ is not balanced we can conclude that the constructed graph contains a vertex of type 3. Finally, we prove that all vertices of type 3 correspond to either a trichromatic triangle or a bichromatic edge with distinct non-consecutive labels and hence k-Polygon Tucker's Lemma follows. Our proof also gives us a reduction of the computational problem associated with k-Polygon Tucker's Lemma to the problem Imbalance-mod-k[#1].

For any simplex $\sigma \in T$ we define $S(\sigma)$ and $\lambda(\sigma)$:

- $S(\sigma) \subseteq [k]$ is the minimal subset of $[k]$ such that σ lies in the cone defined by $\{\boldsymbol{u}_i : i \in S(\sigma)\}$,
- $\lambda(\sigma) = \{\lambda(\boldsymbol{x}) : \boldsymbol{x}$ is a vertex of $\sigma\}$,

and we let $S(\{\mathbf{0}\}) = \emptyset$.

Remark 2. Observe that because T is a refinement of T^*, we have that every simplex $\sigma \in T$ is contained in a cone defined by two consecutive vectors $\boldsymbol{u}_i, \boldsymbol{u}_{i+1}$. Hence, for $\sigma \neq \{\mathbf{0}\}$, $S(\sigma)$ contains either a single number $i \in [k]$ or two consecutive numbers $i, i+1$.

Definition 7 (Happy Simplices)**.** A simplex $\sigma \in T$ is called *happy* if and only if $S(\sigma) \subseteq \lambda(\sigma)$. See also Figure 1 for an example that explains the definition.

We will define a graph G with vertex set $V(G) = T$. In G, we will only add directed edges to the following vertices, which we call *relevant*:

(a) vertices that correspond to a simplex $\sigma \in T$ such that $\sigma \notin \partial T$ and σ is happy,

(b) vertices that correspond to a simplex $\psi \in \partial T$ such that ψ is happy and:

- $\psi = \{u_1\}$, if ψ is 0-dimensional
- $\psi \subseteq \mathsf{conv}(\{\boldsymbol{u}_1, \boldsymbol{u}_2\})$, if ψ is 1-dimensional.

The reason for this distinction between simplices on the boundary and simplices not on the boundary is that we want to identify the symmetric simplices on the boundary as a *super vertex* in order to correctly use a modulo-k argument, as we described in the sketch of the proof.

Remark 3. The rest of the vertices in $V(G)$ that do not correspond to type (a) or (b) simplices can be thought of as having $(\mathsf{out\text{-}degree}) = (\mathsf{in\text{-}degree}) = 0$ or as having a self-loop. In both cases, these vertices are balanced.

We add an edge (v, v') to the graph G only if the simplices σ and σ' that correspond to v and v' are both relevant and one of the following rules applies:

1. **case $\boldsymbol{\sigma \notin \partial T, \sigma' \notin \partial T}$:** We add the edge if $\sigma' \subseteq \sigma$ and the labels of σ' suffice to make σ happy, i.e., $S(\sigma) \subseteq \lambda(\sigma')$,

2. **case $\boldsymbol{\sigma \notin \partial T, \sigma' \in \partial T}$:** Observe that since v' is relevant and $\sigma' \in \partial T$, v' is of type (b). So, instead of checking whether $\sigma' \subseteq \sigma$, we check whether there exists $t \in [k]$ such that $\tau := \theta_k^{(t)}(\sigma') \subseteq \sigma$ and τ suffices to make σ happy, i.e., $S(\sigma) \subseteq \lambda(\tau)$.

Directing the edges. The edge between σ and σ' is directed in the following natural way:

- if σ is 1-dimensional, then σ and σ' lie in $\mathsf{conv}(\{0, \boldsymbol{u}_i\})$ for some i, and the edge is directed "away from 0". Formally, if $\sigma = \{\boldsymbol{z}_0, \boldsymbol{z}_1\}$ and $\sigma' = \{\boldsymbol{z}_1\}$ are connected by an edge, then write $\boldsymbol{z}_0 = \alpha_i u_i$ and $\boldsymbol{z}_1 = \beta_i u_i$. If $\alpha_i - \beta_i > 0$, then the edge is incoming into σ. If $\alpha_i - \beta_i < 0$, then the edge is outgoing out of σ.

- if σ is 2-dimensional, then σ and σ' lie in $\mathsf{conv}(\{0, \boldsymbol{u}_i, \boldsymbol{u}_j\})$ for some i, j with $i - j = \pm 1 \pmod k$. If $j = i + 1$, then the edge is directed such that "we keep label i to our right, and label $j = i + 1$ to our left, when we move in the direction of the edge". If $j = i - 1$, then the edge is directed such that "we keep label $j = i - 1$ to our right, and label i to our left, when we move in the direction of the edge". Formally, if $\sigma = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_2\}$ and $\sigma' = \{\boldsymbol{z}_1, \boldsymbol{z}_2\}$ are connected by an edge, where $\lambda(\boldsymbol{z}_1) = i$ and $\lambda(\boldsymbol{z}_2) = j$, then write $\boldsymbol{z}_0 = \alpha_i \boldsymbol{u}_i + \alpha_j \boldsymbol{u}_j$, $\boldsymbol{z}_1 = \beta_i \boldsymbol{u}_i + \beta_j \boldsymbol{u}_j$ and $\boldsymbol{z}_2 = \gamma_i \boldsymbol{u}_i + \gamma_j \boldsymbol{u}_j$. Construct the matrix

$$M = \begin{bmatrix} \alpha_i - \beta_i & \alpha_i - \gamma_i \\ \alpha_j - \beta_j & \alpha_j - \gamma_j \end{bmatrix}$$

  If $\det M > 0$, then the edge is incoming into σ. If $\det M < 0$, then the edge is outgoing out of σ. Notice that $\det M \neq 0$, because σ is a simplex. Furthermore, note that the determinant of the matrix does not change if we switch both i and j, and $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ (i.e., β and γ). Thus, the direction is well-defined.

If σ' corresponds to a vertex of type (b), then we apply the rule above to σ and τ instead.

Based on the description of the edges in the graph G we can prove the following Lemmas which complete the proof of k-Polygon Tucker's Lemma.

Lemma 3.7. *The vertex that corresponds to the simplex $\{\boldsymbol{0}\} \in T$ has out-degree 1 and in-degree 0.*

Proof. Since $\{\boldsymbol{0}\}$ is a 0-dimensional simplex, it does not have any sub-simplices and hence it can only be connected to a 1-dimensional simplex, i.e., an edge. An edge $\psi \in T$ that is not contained in a linear segment of the form $\mathsf{conv}(\{\boldsymbol{0}, \boldsymbol{u}_i\})$ cannot be a neighbor of $\{\boldsymbol{0}\}$, because it requires two labels to become happy and $\{\boldsymbol{0}\}$ has only one single label. Hence, $\{\boldsymbol{0}\}$ can only be connected to a 1-dimensional simplex $\{\boldsymbol{0}, \boldsymbol{z}_i\}$ contained in $\mathsf{conv}(\{\boldsymbol{0}, \boldsymbol{u}_i\})$ for some i. Observe also that $S(\{\boldsymbol{0}, \boldsymbol{z}_i\}) = \{i\}$, which implies that $S(\{\boldsymbol{0}, \boldsymbol{z}_i\}) = \{\lambda(\boldsymbol{0})\}$. Therefore, there is exactly one 1-dimensional simplex that is connected to $\{\boldsymbol{0}\}$, namely $\{\boldsymbol{0}, \boldsymbol{z}_i\}$, where $i = \lambda(\boldsymbol{0})$.

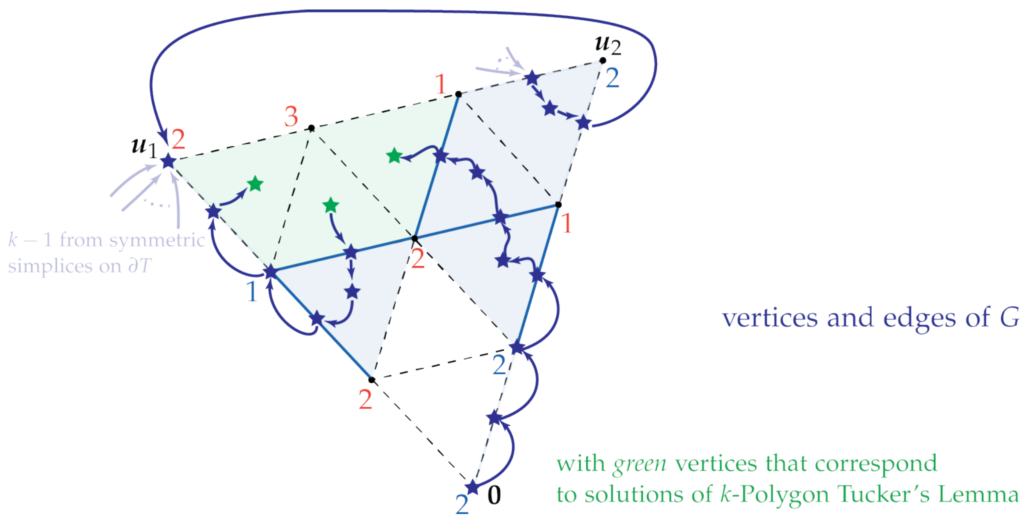

Figure 2: An example of the graph G constructed in the proof of k-Polygon Tucker's Lemma, focused on the cone $\mathsf{conv}(\{\boldsymbol{u}_1, \mathbf{0}, \boldsymbol{u}_2\})$.

Furthermore, since $\{\mathbf{0}\}$ and its neighbor $\{\mathbf{0}, \boldsymbol{z}_i\}$ lie in $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$, the edge is directed away from $\{\mathbf{0}\}$. Formally, if we write $\boldsymbol{z}_i = \alpha_i u_i$ and $\mathbf{0} = \beta_i u_i$, it will hold that $\alpha_i - \beta_i = \alpha_i > 0$. This means that the edge is incoming into $\{\mathbf{0}, \boldsymbol{z}_i\}$ and the lemma follows. $\square$

Lemma 3.8. *Any vertex v in G that is imbalanced modulo-k, i.e.,* $(\mathsf{out\text{-}degree}(v)) \not\equiv (\mathsf{in\text{-}degree}(v)) \pmod k$ *and does not correspond to the simplex $\{\mathbf{0}\}$, corresponds to a simplex σ that is either trichromatic or $\lambda(\sigma) \not\subseteq \{i, i+1\}$ for all $i \in [k]$.*

Proof. For this proof, we consider all three cases for the dimension of the simplex $\sigma^\star$ that corresponds to an imbalanced node of G separately.

Dimension of $\sigma^\star$ is 0. In this case, $\sigma^\star = \{\boldsymbol{z}^\star\}$. It is easy to see that $\sigma^\star$ cannot make happy any 2-dimensional simplex since a 2-dimensional simplex σ has $|S(\sigma)| = 2$. So, $\sigma^\star$ can only be a neighbor of a 1-dimensional simplex ψ. Additionally, $\sigma^\star$ cannot make happy any 1-dimensional ψ such that $|S(\psi)| = 2$. Thus, a neighbor of $\sigma^\star$ should be contained in the segment $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$ where i is such that $\lambda(\sigma^\star) = i$. If $\boldsymbol{z}^\star \neq \boldsymbol{u}_i$, then $\sigma^\star$ is connected to both of its two neighboring 1-dimensional simplices in the segment $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$. Then, since the edges are directed away from $\mathbf{0}$, $\sigma^\star$ has one incoming and one outgoing edge. Formally, let $\{\boldsymbol{z}_0, \boldsymbol{z}^*\}$ and $\{\boldsymbol{z}^*, \boldsymbol{z}_0'\}$ be the two neighboring 1-dimensional simplices, and write $\boldsymbol{z}_0 = \alpha_i u_i$, $\boldsymbol{z}_0' = \alpha_i' u_i$ and $\boldsymbol{z}^* = \beta_i u_i$. It is easy to see that $\alpha_i - \beta_i$ and $\alpha_i' - \beta_i$ always have opposite signs, since $\boldsymbol{z}_0$ and $\boldsymbol{z}_0'$ lie on opposite sides of $\boldsymbol{z}^\star$ on $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$.

If $\boldsymbol{z}^\star = \boldsymbol{u}_i$, then from the definition of relevant vertices of G we have that $\boldsymbol{z}^\star = \boldsymbol{u}_1$ and $\{\boldsymbol{u}_1\}$ is happy, i.e., $\lambda(\boldsymbol{u}_1) = 1$. Thus, by the boundary conditions, for every $t \in [k]$, it holds that $\lambda(\theta_k^{(t)}(\boldsymbol{u}_1)) = t + 1$. In other words, $\lambda(\boldsymbol{u}_i) = i$ for all $i \in [k]$. As a result, it follows that for each $i \in [k]$, $\{\boldsymbol{u}_1\}$ has an edge with the simplex $\{\boldsymbol{u}_i, \boldsymbol{z}_i\}$ which lies in $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$. This holds because $\theta_k^{(i)}(\{\boldsymbol{u}_1\})$ suffices to make $\{\boldsymbol{u}_i, \boldsymbol{z}_i\}$ happy. Clearly, $\theta_k^{(i)}(\{\boldsymbol{u}_1\})$ cannot make any other simplex happy, by the same arguments as above. Finally, note that all the edges are incoming into $\{\boldsymbol{u}_1\}$, because edges are directed away from $\mathbf{0}$ on any $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$. Formally, if we write $\boldsymbol{z}_i = \alpha_i u_i$ and $\boldsymbol{u}_i = \beta_i u_i$, then we always have $\alpha_i - \beta_i = \alpha_i - 1 < 0$, so the edge is outgoing out of $\{\boldsymbol{u}_i, \boldsymbol{z}_i\}$. Thus, in this case, $z^\star$ has k neighbors and all of them with the same direction. Therefore, $z^\star$ is always balanced modulo-k. In conclusion, an imbalanced vertex of G different from $\{\mathbf{0}\}$ cannot have dimension 0.

Dimension of $\sigma^\star$ is 1. First, assume that $\sigma^\star = \{z_1^\star, z_2^\star\}$ belongs to one of the line segments $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$. If additionally $\lambda(z_1^\star) = \lambda(z_2^\star)$, then $\sigma^\star$ is happy if $\lambda(z_1^\star) = \lambda(z_2^\star) = i$. Therefore, $|\lambda(\sigma^\star)| = 1$ and hence $\sigma^\star$ cannot be a neighbor of a 2-dimensional σ since $|S(\sigma)| = 2$ and $\sigma^\star$ cannot make σ happy. So, $\sigma^\star$ has exactly the two neighbors $\{z_1^\star\}$ and $\{z_2^\star\}$ since both of them make $\sigma^\star$ happy. Furthermore, the vertex is balanced, because one of the edges is incoming and the other one is outgoing, since edges are directed away from $\mathbf{0}$ on $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$. Formally, if we write $z_1^\star = \alpha_i u_i$ and $z_2^\star = \alpha_i' u_i$, then $\alpha_i - \alpha_i'$ and $\alpha_i' - \alpha_i$ always have opposite signs.

The next case we consider is when $\sigma^\star \subseteq \mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$ with $i = \lambda(z_1^\star) \neq \lambda(z_2^\star) = j$. If $i - j \neq \pm 1$, then $\sigma^\star$ yields a solution to k-Polygon Tucker, and so is allowed to be imbalanced. On the other hand, if $j = i \pm 1 \pmod k$, $\sigma^\star$ has exactly two neighbors, the simplex $\{z_1^\star\}$, which makes $\sigma^\star$ happy, and the unique 2-dimensional simplex $\sigma = \{z_1^\star, z_2^\star, z_3\}$ that contains $\sigma^\star$ as a face and is contained in the cone $\mathsf{conv}(\{\boldsymbol{u}_i, \mathbf{0}, \boldsymbol{u}_j\})$. Also, one of the edges is incoming and other one outgoing and hence $\sigma^\star$ cannot be imbalanced. Formally, write $z_2^\star = \beta_i u_i$, $z_1^\star = \beta_i' u_i$ and $z_3 = \alpha_i u_i + \alpha_j u_j$. Then, it holds that the edge goes from $\{z_1^\star\}$ to $\{z_1^\star, z_2^\star\}$ if $\beta_i' - \beta_i > 0$, and otherwise in the other direction. Furthermore, the edge goes from $\sigma = \{z_1^\star, z_2^\star\}$ to $\sigma = \{z_1^\star, z_2^\star, z_3\}$, if $\det M > 0$, where we can compute that $\det M = (\alpha_i - \beta_i)\alpha_j - (\alpha_i - \beta_i')\alpha_j = \alpha_j(\beta_i' - \beta_i)$. Since $\alpha_j > 0$, it follows that both expressions have the same sign, and thus $\sigma^\star$ is balanced.

The final case for 1-dimensional $\sigma^\star$ is when $\sigma^\star$ is not contained in any line segment of the form $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i\})$. Let $\sigma^\star$ be contained in the cone $\mathsf{conv}(\{\boldsymbol{u}_i, \mathbf{0}, \boldsymbol{u}_{i+1}\})$. If $\sigma^\star$ is not relevant, then it cannot be imbalanced. To be relevant, and hence happy, it must hold that $\lambda(z_1^\star) = i$ and $\lambda(z_2^\star) = i + 1$. If $\sigma^\star$ is not in the boundary ∂T, then $\sigma^\star$ is the face of exactly two 2-dimensional simplices σ, σ' that are both contained in $\mathsf{conv}(\{\boldsymbol{u}_i, \mathbf{0}, \boldsymbol{u}_{i+1}\})$. Therefore, $\sigma^\star$ makes happy both of them and no other simplex, and consequently it has an edge with both of them and no other edge. Furthermore, one of the edges is incoming and other one is outgoing (from the perspective of $\sigma^\star$), so it cannot be imbalanced. Intuitively, if we let $\sigma = \{z_1^\star, z_2^\star, z_3\}$ and $\sigma' = \{z_1^\star, z_2^\star, z_3'\}$, then z_3 and z_3' lie on opposite sides of the line defined by $\{z_1^\star, z_2^\star\}$. Thus, the basis of $\mathbb{R}^2$ defined by $\{z_3 - z_1^\star, z_3 - z_2^\star\}$ has opposite orientation compared to the basis $\{z_3' - z_1^\star, z_3' - z_2^\star\}$. As a result, $\det M$ will have a different sign for the two edges. For a formal proof of this, we refer to the proof of Theorem 5.3, where we prove a more general version of this fact.

If $\sigma^\star \in \partial T$, then $\sigma^\star$ is relevant only if $\sigma^\star \subseteq \mathsf{conv}(\{\boldsymbol{u}_1, \boldsymbol{u}_2\})$. In this case, $\sigma^\star$ has exactly k neighbors each of which is a 2-dimensional simplex that has as a face one of the k symmetric copies of $\sigma^\star$. Namely, the neighbors of $\sigma^\star$ are $\sigma_1, \ldots, \sigma_k$, where $\theta_k^{(i)}(\sigma^\star)$ makes σ_i happy for all $i \in [k]$. To see that all k edges are incoming or all are outgoing, notice that if we have 1 to our right and 2 to our left when we reach the boundary, then it will hold that we have i on our right and $i+1$ on our left when we reach the boundary at the corresponding position in cone $\mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_i, \boldsymbol{u}_{i+1}\})$. More formally, the sign of the determinant of the matrix $M(\sigma_i, \theta_k^{(i)}(\sigma^\star))$ constructed to determine the direction of the edge with σ_i, will be the same for all $i \in [k]$. For a full formal proof, we again refer to the proof of Theorem 5.3.

Dimension of $\sigma^\star$ is 2. Let $\sigma^\star = \mathsf{conv}(\{z_1^\star, z_2^\star, z_3^\star\})$. Assume that $\sigma^\star$ is contained in the simplex $\mathsf{conv}(\{\boldsymbol{u}_i, \mathbf{0}, \boldsymbol{u}_j\})$, with $j = i \pm 1 \pmod k$ and without loss of generality $\lambda(z_1^\star) = i$, $\lambda(z_2^\star) = j$. If $\lambda(z_3^\star) \notin \{i, j\}$ then $\sigma^\star$ is a trichromatic triangle, and thus it can be imbalanced. Assume that

$\lambda(z_3^\star) \in \{i,j\}$ and without loss of generality $\lambda(z_3^\star) = i$. In this case, $\sigma^\star$ has exactly two neighbors: the 1-dimensional simplices $\psi_1 = \mathsf{conv}(\{z_1^\star, z_2^\star\})$ and $\psi_2 = \mathsf{conv}(\{z_3^\star, z_2^\star\})$. Once again, one edge is incoming and the other one is outgoing, and thus $\sigma^\star$ is balanced. Intuitively, this follows from the fact that if we move with i to our left and j to our right, then we can "enter" the simplex from one side, and "exit" from the other one. More formally, if we write $z_1^\star = \alpha_i u_i + \alpha_j u_j$, $z_2^\star = \beta_i u_i + \beta_j u_j$ and $z_3^\star = \alpha_i' u_i + \alpha_j' u_j$, then it holds that:

$$\det \begin{bmatrix} \alpha_i - \beta_i & \alpha_i - \alpha_i' \\ \alpha_j - \beta_j & \alpha_j - \alpha_j' \end{bmatrix} = -\det \begin{bmatrix} \alpha_i' - \beta_i & \alpha_i' - \alpha_i \\ \alpha_j' - \beta_j & \alpha_j' - \alpha_j \end{bmatrix}$$

by using standard rules about the determinant.

Hence, the only imbalanced vertices different from $\{\mathbf{0}\}$ correspond to either trichromatic simplices or simplices such that $\lambda(\sigma) \not\subseteq \{i, i+1\}$ for all $i \in [k]$. $\square$

Finally, from the definition of the graph G and Lemma 3.7, we have that there has to be a vertex in G that is imbalanced $\pmod k$ and different from $\{\mathbf{0}\}$. By Lemma 3.8, any such vertex proves the validity of k-Polygon Tucker's Lemma.

3.1.3 Computational Problems and Containment in PPA-k[#1]

In this section, we define the computational problems associated with k-Polygon Tucker's Lemma and the k-Polygon Borsuk-Ulam Theorem, which we call k-polygon-Tucker and k-polygon-Borsuk-Ulam respectively. Following the ideas presented in Section 3.1.2, we show that k-polygon-Tucker is in PPA-k[#1]. The membership of k-polygon-Tucker in PPA-k[#1] combined with the results of Section 3.1.1 implies that k-polygon-Borsuk-Ulam is also in PPA-k[#1].

To define the computational problem associated with k-Polygon Tucker's Lemma we need succinct access to the labels $\lambda(\boldsymbol{x})$ for the vertices $\boldsymbol{x}$ in a nice triangulation T of W_k. This succinct access resembles the one in the definition of the computational version of the original Borsuk-Ulam Theorem [Aisenberg et al., 2020, Papadimitriou, 1994] and the computational version of Brouwer's Fixed Point Theorem [Papadimitriou, 1994]. To define this succinct access, we fix for any $m \in \mathbb{N}$ a triangulation $T(m)$ with diameter (i.e., max distance between two vertices of a simplex) at most $1/2^m$, where we can refer to a simplex in $T(m)$ using $O(m+k)$ bits. For our case this triangulation will be the following.

Definition 8 (Edge Parallel Triangulation)**.** For every $m \in \mathbb{N}$, we define $\widehat{T}(m)$ to be the following nice triangulation of W_k. Starting from $T^\star$ (see Definition 6) we define a simplicial complex $\widehat{T}_i(m)$ of every simplex $\sigma_i^\star = \mathsf{conv}(\{\boldsymbol{u}_i, \mathbf{0}, \boldsymbol{u}_{i+1 \pmod k}\})$ and then set $\widehat{T}(m) = \cup_{i\in[k]} \widehat{T}_i(m)$. To define $\widehat{T}_i(m)$, we divide the edges $\psi_1^i = \mathsf{conv}(\{\boldsymbol{u}_i, \mathbf{0}\})$, $\psi_2^i = \mathsf{conv}(\{\boldsymbol{u}_i, \boldsymbol{u}_{i+1 \pmod k}\})$ and $\psi_3^i = \mathsf{conv}(\{\mathbf{0}, \boldsymbol{u}_{i+1 \pmod k}\})$ of $\sigma_i^\star$ equally into 2^{m+1} intervals. Then, from any endpoint of the subintervals of the edge ψ_2^i we consider the lines that are parallel to either ψ_1^i or ψ_3^i and from any endpoint of the subintervals in ψ_1^i and ψ_3^i we consider the line that is parallel to ψ_2^i, as shown in Figure 2. We define $\widehat{T}_i(m)$ to be the set of simplices that are created by these lines and lie inside $\sigma_i^\star$. Namely, the intersection points and endpoints of subintervals are the 0-dimensional simplices in $\widehat{T}_i(m)$, the line segments and the triangles between 0-dimensional simplices are also simplices in $\widehat{T}_i(m)$. It is simple to see that $\widehat{T}(m)$ is nice; we call $\widehat{T}(m)$ an *edge parallel triangulation* of W_k.

For the edge parallel triangulation $\widehat{T}(m)$ of W_k the following facts are easy to verify.

Fact 3.9. *The diameter of $\widehat{T}(m)$ is at most $1/2^m$.*

Fact 3.10. *We can indicate uniquely a simplex $\sigma_i^\star$ using $\lceil \log(k) \rceil$ bits. Then, using $2m+3$ bits we can indicate uniquely a combination of two lines: (1) one of the $2 \cdot 2^{m+1}$ lines that are parallel to either ψ_1^i or ψ_3^i, and (2) one of the 2^{m+1} lines that are parallel to ψ_2^i. This combination uniquely determines a point $\boldsymbol{x} \in \mathbb{R}^2$ and we can efficiently check whether this point belongs to $\sigma_i^\star$ or not. Of course this way the points that lie on the rays from $\mathbf{0}$ to $\boldsymbol{u}_i$ have two different representations but we can easily resolve this discrepancy by choosing as valid only the representation that is lexicographically first. Hence, we can uniquely determine any vertex in $\widehat{T}(m)$ with $b = \lceil \log(k) \rceil + 2m + 3$ bits.*

Fact 3.11. *Given a set of binary vectors $A = \{\boldsymbol{a}_i\}_i$ where $\boldsymbol{a}_i \in \{0,1\}^b$, there exists an efficient procedure that determines whether $\mathsf{conv}(A)$ is a simplex of $\widehat{T}(m)$ or not.*

Because of Fact 3.10, we can assume that the labeling λ of $\widehat{T}(m)$ is given via a circuit $\mathcal{L}$ with $b = \lceil \log(k) \rceil + 2m + 3$ input bits and $\lceil \log(k) \rceil$ output bits. The input to the circuit $\mathcal{L}$ is the representation of a potential vertex $\boldsymbol{x}$ in $\widehat{T}(m)$ according to Fact 3.10 and the output is the label $\lambda(x) \in [k]$ of this vertex $\boldsymbol{x}$. Observe that Fact 3.10 also guarantees that it is easy to check whether an input $\boldsymbol{a} \in \{0,1\}^b$ to the circuit $\mathcal{L}$ corresponds to a valid vertex $\boldsymbol{x} \in \widehat{T}(m)$. We are now ready to define the total search problem that is associated with k-Polygon Tucker's Lemma.

k-POLYGON-TUCKER

INPUT: A circuit $\mathcal{L} : \{0,1\}^b \to \{0,1\}^{\lceil \log(k) \rceil}$, with $b = \lceil \log(k) \rceil + 2m + 3$.

OUTPUT: One of the following.

1. Two binary vectors $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \{0,1\}^b$ such that $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ correspond to vertices $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ on $\partial\widehat{T}(m)$ with $\boldsymbol{x}_2 = \theta_k(\boldsymbol{x}_1)$, but $\mathcal{L}(\boldsymbol{a}_2) \neq \mathcal{L}(\boldsymbol{a}_1) + 1 \pmod{k}$.
2. Three binary vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in \{0,1\}^b$ such that the simplex $\sigma = \mathsf{conv}(\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\})$ belongs to $\widehat{T}(m)$ and all the labels $\mathcal{L}(\boldsymbol{a}_1)$, $\mathcal{L}(\boldsymbol{a}_2)$, $\mathcal{L}(\boldsymbol{a}_3)$ are different from each other.
3. Two binary vectors $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \{0,1\}^b$ such that the edge $\psi = \mathsf{conv}(\{\boldsymbol{a}_1, \boldsymbol{a}_2\})$ belongs to $\widehat{T}(m)$ and labels $\mathcal{L}(\boldsymbol{a}_1)$, $\mathcal{L}(\boldsymbol{a}_2)$ are different and they satisfy $\mathcal{L}(\boldsymbol{a}_1) - \mathcal{L}(\boldsymbol{a}_2) \neq \pm 1 \pmod{k}$.

Lemma 3.12. *It holds that k-POLYGON-TUCKER is in* PPA-k[#1].

Proof. This lemma follows from the proof of k-Polygon Tucker's Lemma that we presented in Section 3.1.2. We only need to add the description of the circuits $\mathcal{S}$, $\mathcal{P}$ that define the graph G constructed in the proof. Then, using these circuits we reduce k-POLYGON-TUCKER to IMBALANCE-MOD-k[#1]. As we showed in Lemma 3.8, every solution to the resulting instance of IMBALANCE-MOD-k[#1] corresponds to a solution of k-POLYGON-TUCKER. Hence, k-POLYGON-TUCKER is in PPA-k[#1].

Since the vertices of the graph G correspond to simplices in $\widehat{T}(m)$ and the maximum degree of any node in G is k, we define the circuits $\mathcal{S}$, $\mathcal{P}$ to have $3 \cdot b$ binary inputs and $k \cdot (3b)$ outputs, hence $\mathcal{S} : \{0,1\}^{3b} \to (\{0,1\}^{3b})^k$ and $\mathcal{P} : \{0,1\}^{3b} \to (\{0,1\}^{3b})^k$. For the construction of the circuits, both $\mathcal{S}$ and $\mathcal{P}$ first check whether an input $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ is *valid* or not. An input $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ is valid in the following cases.

▷ If all a_1, a_2, a_3 are different, then (a_1, a_2, a_3) is valid if a_1, a_2, a_3 are in lexicographical order and the simplex $\mathsf{conv}(\{x_1, x_2, x_3\})$, where x_i is the point in $\mathbb{R}^2$ that corresponds to a_i according to Fact 3.10, is a valid simplex of $\widehat{T}(m)$. Observe that we can check this using a polynomial size circuit by Fact 3.11.

▷ If $\{a_1, a_2, a_3\}$ has two different vectors, then (a_1, a_2, a_3) is valid if the lexicographically first is a_1, the lexicographically second is $a_2 = a_3$ and the edge $\mathsf{conv}(\{x_1, x_2\})$, where x_i is the point in $\mathbb{R}^2$ that corresponds to a_i according to Fact 3.10, is a valid simplex of $\widehat{T}(m)$. Observe that we can check this using a polynomial size circuit by Fact 3.11.

▷ If $a_1 = a_2 = a_3$, then (a_1, a_2, a_3) is valid if the point x_1 in $\mathbb{R}^2$ that corresponds to a_1 according to Fact 3.10 is a valid vertex of $\widehat{T}(m)$. Observe that we can check this using a polynomial size circuit by Fact 3.11.

If input (a_1, a_2, a_3) is not valid, then both circuits $\mathcal{S}$ and $\mathcal{P}$ output (a_1, a_2, a_3) concatenated with itself k times. On the other hand, if (a_1, a_2, a_3) is valid, then both circuits check whether this input corresponds to a relevant simplex as we defined in Section 3.1.2. It is easy to see that checking relevance can be done efficiently. Again, if (a_1, a_2, a_3) is not relevant, both circuits $\mathcal{S}$ and $\mathcal{P}$ output (a_1, a_2, a_3) concatenated with itself k times. Finally, if (a_1, a_2, a_3) is valid and relevant, then we define the edges of the corresponding vertex in G as described in Section 3.1.2. From the construction of G, it follows that the successors and the predecessors can be computed efficiently. If the vertex in G that corresponds to (a_1, a_2, a_3) has less than k incoming or outgoing edges, then we repeat the lexicographically last neighboring vertex enough times such that the number of bits in the output of both $\mathcal{S}$ and $\mathcal{P}$ is $k \cdot (3b)$.

Using our analysis in Section 3.1.2, it follows that the above construction of $\mathcal{S}$ and $\mathcal{P}$ defines a reduction from k-POLYGON-TUCKER to IMBALANCE-MOD-k[#1]. □

We now define the computational problem associated with the k-Polygon Borsuk-Ulam Theorem. For this computational problem we need a representation of the continuous function f that is the input to the k-Polygon Borsuk-Ulam Theorem. Following standard techniques in the literature, we use arithmetic circuits with gates $\times\zeta$ (multiplication by a constant), $+$, $-$, $<$, min, and max and rational constants to define this function.

<u>k-POLYGON-BORSUK-ULAM</u>

INPUT: An arithmetic circuit $\mathcal{C} : B^2 \to \mathbb{R}^2$, an accuracy parameter $\varepsilon > 0$ and a Lipschitz-constant L.

OUTPUT: One of the following.

1. A point $\mathbf{x} \in S^1$ such that $\|\mathcal{C}(\tilde{\theta}_k(\mathbf{x})) - \tilde{\theta}_k(\mathcal{C}(\mathbf{x}))\| \geq \eta(\varepsilon, k) := \varepsilon/8k^4$.
2. Two points $\mathbf{x}, \mathbf{y} \in B^2$ such that $\|\mathcal{C}(\mathbf{x}) - \mathcal{C}(\mathbf{y})\| > L\,\|\mathbf{x} - \mathbf{y}\|$.
3. A point $\mathbf{x}^* \in B^2$ such that $\|\mathcal{C}(\mathbf{x}^*)\| \leq \varepsilon$.

The first type of solution corresponds to a violation of the boundary conditions. Since we cannot compute θ_k exactly, we let $\tilde{\theta}_k$ denote a function that computes θ_k with error at most $\xi(\varepsilon, L, k) := \frac{\varepsilon}{2^7 L k^4}$. The second type of solution corresponds to a violation of L-Lipschitz-continuity. Note that the function computed by the circuit $\mathcal{C}$ might not be continuous, because of the comparison gate. Thus we add this extra violation, which ensures that the function is Lipschitz-continuous. This allows us to relate this problem to k-POLYGON-TUCKER (and in particular to show that it always has a solution).

Lemma 3.13. *The problem k-POLYGON-BORSUK-ULAM reduces to the problem k-POLYGON-TUCKER. Therefore, k-POLYGON-BORSUK-ULAM is in* PPA-k.

Proof. This result is obtained by following the idea of the proof of Lemma 3.6, and constructing a labeling $\mathcal{L}$ that uses the circuit $\mathcal{C}$ as a sub-routine to compute the labels of the corresponding k-POLYGON-TUCKER instance. The details are a bit tricky because we have to account for small errors in various computations.

In more detail, the regular procedure for picking a label at some point is to first compute an approximate value of its coordinates and then based on this to compute an approximate value of the function. However, this might introduce bogus boundary condition violations, even if the function perfectly satisfies the boundary conditions. We can resolve this as follows. For any vertex $\mathbf{x}$ that lies on the boundary, but not between $\mathbf{u}_1$ and $\mathbf{u}_2$, we first check how far away the label obtained through the regular procedure is from the label that satisfies the boundary condition. If it happens that the regular procedure is close to outputting the label that does not violate the boundary conditions then we enforce the output of this label and ignore the output of the regular procedure. Otherwise, we output the label of the regular procedure. Then, we can show that from any solution of k-Polygon Tucker we can either extract an approximate zero of $\mathcal{C}$, or a violation of the boundary conditions of $\mathcal{C}$ (in their approximate version), or a violation of Lipschitz continuity.

We will use the triangulation $\widehat{T}(m)$ where m is picked sufficiently small so that the diameter of the triangulation when mapped to B^2 by the homeomorphism is at most $\delta = \frac{\varepsilon}{2^7 L k^4}$. The Boolean circuit computing $\mathcal{L}$ performs the following operations. Recall that the input to the circuit is the bits representing the index of the vertex $\mathbf{x} \in W_k$ of the triangulation, as per Fact 3.10.

1. Let $\mathbf{y} \in B^2$ denote the image of $\mathbf{x}$ under the homeomorphism used in the proof of Lemma 3.6. Compute an approximation $\tilde{\mathbf{y}}$ of $\mathbf{y}$ with error at most $\eta_1 = \delta = \frac{\varepsilon}{2^7 L k^4}$, i.e., $\|\mathbf{y} - \tilde{\mathbf{y}}\| \leq \eta_1$. Note this is done in two steps, first compute an approximate value for the coordinates of $\mathbf{x}$, which is given through its index, and compute an approximate value of the homeomorphism given the approximate value of the coordinates of $\mathbf{x}$. These steps can be done efficiently by using standard techniques [Brent, 1976] so that the total approximation error is η_1.

2. Compute $\mathcal{C}(\tilde{\mathbf{y}})$ exactly.

3. For each $i \in [k]$, compute an estimate $v_i(\tilde{\mathbf{y}})$ of the inner product $\langle \mathcal{C}(\tilde{\mathbf{y}}), \mathbf{u}_i \rangle$ with error at most $\eta_2 = \varepsilon / 64k^4$, where the approximation error is introduced in computing the coordinates of $\mathbf{u}_i$.

4. If $\mathbf{x} \notin \partial\widehat{T}(m)$ or if $\mathbf{x} \in \mathrm{conv}(\{\mathbf{u}_1, \mathbf{u}_2\}) \setminus \{\mathbf{u}_2\}$, then output the label $\mathrm{argmax}_i\, v_i(\tilde{\mathbf{y}})$ (break ties arbitrarily but deterministically, e.g., lexicographically).

5. Otherwise, there exists $\ell \in [k-1]$ and a vertex $\mathbf{x}' \in \mathrm{conv}(\{\mathbf{u}_1, \mathbf{u}_2\}) \setminus \{\mathbf{u}_2\}$, such that $\mathbf{x} = \theta_k^\ell(\mathbf{x}')$. In that case, compute $r = \max_i v_i(\tilde{\mathbf{y}})$ and $j = \mathcal{L}(\mathbf{x}') + \ell \pmod k$. Let $\eta_3 = \frac{\varepsilon}{16k^4}(3+k)$. If $r - v_j(\tilde{\mathbf{y}}) \leq \eta_3$ then output $\mathcal{L}(\mathbf{x}') + \ell \pmod k$, otherwise output $\mathrm{argmax}_i\, v_i(\tilde{\mathbf{y}})$ (again break ties arbitrarily but deterministically, e.g., lexicographically).

We will use the following useful invariant: if vertex $\mathbf{x}$ has label j, then

$$\langle \mathcal{C}(\tilde{\mathbf{y}}), \mathbf{u}_j \rangle \geq \max_i \langle \mathcal{C}(\tilde{\mathbf{y}}), \mathbf{u}_i \rangle - 2\eta_2 - \eta_3.$$

This follows from our labeling procedure.

We begin by considering standard solutions of k-POLYGON-TUCKER and distinguish between the cases $k = 3$ and $k > 3$. The other type of solution, namely boundary violations are treated for any $k \geq 3$ at the end of the proof.

$\boldsymbol{k = 3}$. In this case a standard solution to $\mathcal{L}$ consists of a simplex with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ that have labels $1, 2, 3$ respectively. Let $\tilde{\mathbf{y}}_1, \tilde{\mathbf{y}}_2, \tilde{\mathbf{y}}_3$ denote the corresponding points computed in the aforementioned step 1. of the computation of $\mathcal{L}$ with input $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ respectively. Assume that $\|\mathcal{C}(\tilde{\mathbf{y}}_i)\| \geq \varepsilon$ for $i = 1, 2, 3$, since we have found a solution otherwise. Using elementary linear algebra, it is easy to show that for any $\tilde{\mathbf{y}} \in B^2$ with $\|\mathcal{C}(\tilde{\mathbf{y}})\| \geq \varepsilon$, it holds that $\max_i \langle \mathcal{C}(\tilde{\mathbf{y}}), \mathbf{u}_i \rangle \geq \varepsilon/2$ and $\min_i \langle \mathcal{C}(\tilde{\mathbf{y}}), \mathbf{u}_i \rangle \leq -\varepsilon/2$.

Let $j = \operatorname{argmin}_i \langle \mathcal{C}(\tilde{\mathbf{y}}_1), \mathbf{u}_i \rangle$. Then it follows that $\langle \mathcal{C}(\tilde{\mathbf{y}}_1), \mathbf{u}_j \rangle \leq -\varepsilon/2$, which implies that $\langle \mathcal{C}(\tilde{\mathbf{y}}_j), \mathbf{u}_j \rangle \leq -\varepsilon/4$, since

$$|\langle \mathcal{C}(\tilde{\mathbf{y}}_1), \mathbf{u}_j \rangle - \langle \mathcal{C}(\tilde{\mathbf{y}}_j), \mathbf{u}_j \rangle| \leq \|\mathcal{C}(\tilde{\mathbf{y}}_1) - \mathcal{C}(\tilde{\mathbf{y}}_j)\| \leq L\|\tilde{\mathbf{y}}_1 - \tilde{\mathbf{y}}_j\| \leq L(\delta + 2\eta_1) \leq \varepsilon/4$$

unless $\tilde{\mathbf{y}}_1$ and $\tilde{\mathbf{y}}_j$ yield a violation of L-Lipschitz continuity of $\mathcal{C}$, in which case we have found a solution. Now, recall that $\mathbf{x}_j$ has label j. By the invariant, it must hold that $\langle \mathcal{C}(\tilde{\mathbf{y}}_j), \mathbf{u}_j \rangle \geq \max_i \langle \mathcal{C}(\tilde{\mathbf{y}}_j), \mathbf{u}_i \rangle - 2\eta_2 - \eta_3$. But since $\max_i \langle \mathcal{C}(\tilde{\mathbf{y}}_j), \mathbf{u}_i \rangle \geq \varepsilon/2$, this means that $\langle \mathcal{C}(\tilde{\mathbf{y}}_j), \mathbf{u}_j \rangle \geq \varepsilon/2 - 2\eta_2 - \eta_3$, which contradicts the fact that $\langle \mathcal{C}(\tilde{\mathbf{y}}_j), \mathbf{u}_j \rangle \leq -\varepsilon/4$.

$\boldsymbol{k > 3}$. In this case a solution to $\mathcal{L}$ consists of an edge $\mathbf{x}_1, \mathbf{x}_2$ of the triangulation such that the two vertices have labels j_1, j_2 that differ by more than one. Assume that $\|\mathcal{C}(\tilde{\mathbf{y}}_1)\| \geq \varepsilon$ and $\|\mathcal{C}(\tilde{\mathbf{y}}_2)\| \geq \varepsilon$. Note that $\tilde{\mathbf{y}}_1$ lies in some cone $\operatorname{conv}(\{\mathbf{0}, \mathbf{u}_t, \mathbf{u}_{t+1}\})$. By elementary linear algebra, one can show that

$$\max_{i \in \{t, t+1\}} \langle \mathcal{C}(\tilde{\mathbf{y}}_1), \mathbf{u}_i \rangle - \max_{i \notin \{t, t+1\}} \langle \mathcal{C}(\tilde{\mathbf{y}}_1), \mathbf{u}_i \rangle \geq \frac{\varepsilon}{2}(1 - \cos(2\pi/k)) \geq \frac{\varepsilon}{k^3} \tag{1}$$

Since $\frac{\varepsilon}{k^3} \geq \eta_3 + 2\eta_2$, it follows that the label j_1 of x_1 must be one of t or $t + 1$. As a result, $j_2 \notin \{t, t+1\}$, since it cannot be adjacent to j_1. Furthermore, since j_1 was picked as the label of x_1, by the invariant it holds that $\langle \mathcal{C}(\tilde{\mathbf{y}}_1), \mathbf{u}_{j_1} \rangle \geq \max_i \langle \mathcal{C}(\tilde{\mathbf{y}}_1), \mathbf{u}_i \rangle - 2\eta_2 - \eta_3$. Together with equation (1), it follows that

$$\langle \mathcal{C}(\tilde{\mathbf{y}}_1), \mathbf{u}_{j_1} \rangle - \langle \mathcal{C}(\tilde{\mathbf{y}}_1), \mathbf{u}_{j_2} \rangle \geq \frac{\varepsilon}{k^3} - 2\eta_2 - \eta_3.$$

On the other hand, since x_2 has label j_2, this means that

$$\langle \mathcal{C}(\tilde{\mathbf{y}}_2), \mathbf{u}_{j_1} \rangle - \langle \mathcal{C}(\tilde{\mathbf{y}}_2), \mathbf{u}_{j_2} \rangle \leq 2\eta_2 + \eta_3.$$

As a result, it must be that $\|\mathcal{C}(\tilde{\mathbf{y}}_1) - \mathcal{C}(\tilde{\mathbf{y}}_2)\| \geq \frac{\varepsilon}{2k^3} - 2\eta_2 - \eta_3$. Note that this is a violation of Lipschitz continuity, since normally we should have

$$\|\mathcal{C}(\tilde{\mathbf{y}}_1) - \mathcal{C}(\tilde{\mathbf{y}}_2)\| \leq L\|\tilde{\mathbf{y}}_1 - \tilde{\mathbf{y}}_2\| \leq L(\delta + 2\eta_1)$$

and this quantity is smaller than $\frac{\varepsilon}{2k^3} - 2\eta_2 - \eta_3$.

Boundary violations. Let $\mathbf{x}$ be a violation of the boundary conditions, i.e., the label at vertex $\mathbf{x}$ is j_1, and the label at $\theta_k(\mathbf{x})$ is $j_2 \neq j_1 + 1 \pmod{k}$. This means that at least one of the two vertices did not obtain its "intended" label. Without loss of generality, assume that $\mathbf{x}$ has not obtained its intended label. Let $\ell \in [k-1]$ and $\mathbf{x}_0 \in \mathrm{conv}(\{\mathbf{u}_1, \mathbf{u}_2\}) \setminus \{\mathbf{u}_2\}$ be such that $\theta_k^\ell(\mathbf{x}_0) = \mathbf{x}$. Since $\mathbf{x}$ did not obtain its "intended" label, it follows that $\max_i v_i(\tilde{\mathbf{y}}) - v_j(\tilde{\mathbf{y}}) > \eta_3$, where $\tilde{\mathbf{y}}$ is the corresponding point in B^2 computed in step 1. This implies that

$$\max_i \langle \mathcal{C}(\tilde{\mathbf{y}}), \mathbf{u}_i \rangle - \langle \mathcal{C}(\tilde{\mathbf{y}}), \mathbf{u}_j \rangle > \eta_3 - 2\eta_2.$$

On the other hand, since $\mathbf{x}_0 \in \mathrm{conv}(\{\mathbf{u}_1, \mathbf{u}_2\}) \setminus \{\mathbf{u}_2\}$ has label $j - \ell \pmod{k}$, it holds that

$$\max_i \langle \mathcal{C}(\tilde{\mathbf{y}}_0), \mathbf{u}_i \rangle - \langle \mathcal{C}(\tilde{\mathbf{y}}_0), \mathbf{u}_{j-\ell \pmod{k}} \rangle \leq 2\eta_2.$$

where $\tilde{\mathbf{y}}_0$ is the corresponding point of $\mathbf{x}_0$ in B^2 computed in step 1. By noting that

$$\max_i \langle \mathcal{C}(\tilde{\mathbf{y}}_0), \mathbf{u}_i \rangle - \langle \mathcal{C}(\tilde{\mathbf{y}}_0), \mathbf{u}_{j-\ell \pmod{k}} \rangle = \max_i \langle \theta_k^\ell(\mathcal{C}(\tilde{\mathbf{y}}_0)), \mathbf{u}_i \rangle - \langle \theta_k^\ell(\mathcal{C}(\tilde{\mathbf{y}}_0)), \mathbf{u}_j \rangle$$

we thus obtain that there exists $i \in [k]$ such that

$$|\langle \mathcal{C}(\tilde{\mathbf{y}}), \mathbf{u}_i \rangle - \langle \theta_k^\ell(\mathcal{C}(\tilde{\mathbf{y}}_0)), \mathbf{u}_i \rangle| > (\eta_3 - 4\eta_2)/2.$$

This implies that $\|\mathcal{C}(\tilde{\mathbf{y}}) - \theta_k^\ell(\mathcal{C}(\tilde{\mathbf{y}}_0))\| \geq (\eta_3 - 4\eta_2)/2$. Since

$$\mathcal{C}(\tilde{\mathbf{y}}) - \theta_k^\ell(\mathcal{C}(\tilde{\mathbf{y}}_0)) = \sum_{i=0}^{\ell-1} \theta_k^i \mathcal{C}(\theta_k^{-i}(\tilde{\mathbf{y}})) - \theta_k^{i+1}(\mathcal{C}(\theta_k^{\ell-i-1}(\tilde{\mathbf{y}}_0)))$$

it follows that there exists $i \in \{0, 1, \ldots, \ell - 1\}$ such that

$$\|\theta_k^i \mathcal{C}(\theta_k^{-i}(\tilde{\mathbf{y}})) - \theta_k^{i+1}(\mathcal{C}(\theta_k^{\ell-i-1}(\tilde{\mathbf{y}}_0)))\| \geq (\eta_3 - 4\eta_2)/(2k).$$

Now let $\mathbf{x}^* = \theta_k^{\ell-i-1}(\mathbf{x}_0)$. Then $\tilde{\mathbf{y}}^*$ yields a violation of the boundary conditions for $\mathcal{C}$ by noting that

$$\begin{aligned}
\|\mathcal{C}(\tilde{\theta}_k(\tilde{\mathbf{y}}^*)) - \tilde{\theta}_k(\mathcal{C}(\tilde{\mathbf{y}}^*))\| &\geq (\eta_3 - 4\eta_2)/(2k) - \|\mathcal{C}(\tilde{\theta}_k(\tilde{\mathbf{y}}^*)) - \mathcal{C}(\theta_k(\tilde{\mathbf{y}}^*))\| - \|\tilde{\theta}_k(\mathcal{C}(\tilde{\mathbf{y}}^*)) - \theta_k(\mathcal{C}(\tilde{\mathbf{y}}^*))\| \\
&\geq (\eta_3 - 4\eta_2)/(2k) - L(2\eta_1 + \xi(\varepsilon, L, k)) - \xi(\varepsilon, L, k)
\end{aligned}$$

unless we find a violation of Lipschitz continuity. Indeed, note that $(\eta_3 - 4\eta_2)/(2k) - L(2\eta_1 + \xi(\varepsilon, L, k)) - \xi(\varepsilon, L, k) \geq \eta(\varepsilon, k)$. □

3.2 *k*-Polygon Borsuk-Ulam and *k*-Polygon Tucker are PPA-*k*[#1]-hard

In this section we show the PPA-*k*[#1]-hardness of both *k*-POLYGON-TUCKER and *k*-POLYGON-BORSUK-ULAM. We start by observing that using the ideas from Section 3.1.1 we can prove that *k*-POLYGON-TUCKER is reducible to *k*-POLYGON-BORSUK-ULAM. Then we show that *k*-POLYGON-TUCKER is PPA-*k*[#1]-hard. Finally the results of this section together with the results of the previous Section 3.1.3 imply that both *k*-POLYGON-TUCKER and *k*-POLYGON-BORSUK-ULAM are PPA-*k*[#1]-complete.

Lemma 3.14. *The problem k-POLYGON-TUCKER reduces to the problem k-POLYGON-BORSUK-ULAM.*

Proof. This Lemma follows from the proof of Lemma 3.5 combined with standard arguments on how to construct an arithmetic circuit approximately computing the piecewise linear function $g : B^2 \to \mathbb{R}^2$ described in Lemma 3.5. Recall that the function g is constructed by interpolating within simplices of the triangulation using the labels given by $\mathcal{L}$. Given a precision parameter γ and the Boolean circuit $\mathcal{L}$, we can construct an arithmetic circuit $\mathcal{C}$ that computes the function g with error at most γ, using standard techniques [Etessami and Yannakakis, 2010, Daskalakis and Papadimitriou, 2011, Goldberg and Hollender, 2019]. To be more precise, we can invoke Theorem E.2 from [Fearnley et al., 2020] by observing that the function g is polynomially approximately computable via standard numerical analysis techniques Brent [1976]. Furthermore, the function computed by $\mathcal{C}$ will be Lipschitz-continuous with a Lipschitz constant $\tilde{L}$ that has bit-size polynomial in m and $\log(1/\gamma)$ (where m is the parameter of the triangulation used by $\mathcal{L}$).

For any $\mathbf{x} \in B^2$ that does not lie in a simplex that is a solution of $\mathcal{L}$, it must hold that $\|g(\mathbf{x})\| \geq 1/2$. This can easily be shown by taking into account that at most two labels (that are also adjacent) are used for the interpolation in that case. As a result, as long as we ensure that $1/2 - \gamma > \varepsilon$, any point with $\|\mathcal{C}(\mathbf{x})\| \leq \varepsilon$ will yield a solution of the k-POLYGON-TUCKER instance.

It remains to show that no bogus boundary condition violations occur. Note that unless the simplex containing $\mathbf{x} \in \partial B^2$ yields a boundary condition violation for $\mathcal{L}$, it must hold that $g(\theta_k(\mathbf{x})) = \theta_k(g(\mathbf{x}))$ (as shown in Lemma 3.5). Thus, it follows that

$$\begin{aligned}\|\mathcal{C}(\tilde{\theta}_k(\mathbf{x})) - \tilde{\theta}_k(\mathcal{C}(\mathbf{x}))\| &\leq \|\mathcal{C}(\theta_k(\mathbf{x})) - \theta_k(\mathcal{C}(\mathbf{x}))\| + \|\mathcal{C}(\tilde{\theta}_k(\mathbf{x})) - \mathcal{C}(\theta_k(\mathbf{x}))\| + \|\tilde{\theta}_k(\mathcal{C}(\mathbf{x})) - \theta_k(\mathcal{C}(\mathbf{x}))\| \\ &\leq 2\gamma + \tilde{L} \cdot \xi(\varepsilon, L, k) + \xi(\varepsilon, L, k).\end{aligned}$$

Note that we want this quantity to be strictly less than $\eta(\varepsilon, k)$. Thus, we pick $\varepsilon = 1/4$ and then set γ and L so that

- $L \geq \tilde{L}$,
- $\gamma < 1/2 - \varepsilon$,
- $2\gamma + \tilde{L} \cdot \xi(\varepsilon, L, k) + \xi(\varepsilon, L, k) < \eta(\varepsilon, k)$,

which can easily be achieved by picking γ sufficiently small and L sufficiently large. Note that since $\mathcal{C}$ is $\tilde{L}$-Lipschitz-continuous, it will also be L-Lipschitz-continuous. □

Now we present our main proposition in this section.

Proposition 3.15. *For all $k \geq 3$, k-POLYGON-TUCKER is* PPA-k[#1]-hard.

Recall that PPA-k[#1] $= \cap_{p \in PF(k)}$PPA-p, where $PF(k)$ denotes the set of prime factors of k.

Proof. To prove this hardness result we reduce from BIPARTITE-MOD-k[#1] (see Section 2.2 for the formal definition). Recall that in this problem we are given a bipartite graph on the set of nodes $A \cup B$, where $A = \{0\} \times \{0,1\}^n$ and $B = \{1\} \times \{0,1\}^n$, such that the node $00^n \in A$ has degree 1. The goal is to find any other node that has degree $\neq 0 \mod k$. All nodes have degree in $\{0, 1, \ldots, k\}$ and we can assume (see e.g., [Hollender, 2019, Section 4.2]) that the circuit C which implicitly represents the bipartite graph is consistent, i.e., for all x, y we have $y \in C(x)$ iff $x \in C(y)$.

Consider the regular k-polygon W_k in $\mathbb{R}^2$ with the edge parallel triangulation (Definition 8). For any $i \in \mathbb{Z}_k$ let $R(i, i+1)$ denote the triangle with endpoints $\{\mathbf{0}, \mathbf{u}_i, \mathbf{u}_{i+1}\}$. In this proof we refer to *nodes* of the BIPARTITE-MOD-k[#1] instance and to *vertices* of the triangulation of W_k.

To every node $x \in (A \setminus \{00^n\}) \cup B$ we associate a distinct interval on the outer boundary of $R(1,2)$, i.e., on the edge $\mathsf{conv}(\{\mathbf{u}_1, \mathbf{u}_2\})$. This interval, which we denote by $K_1(x)$, is picked such that it covers $4k$ vertices of the triangulation. It is easy to see that we can pick the triangulation to be fine enough so that there are indeed enough vertices on $\mathsf{conv}(\{\mathbf{u}_1, \mathbf{u}_2\})$ to associate a distinct interval of $4k$ vertices to each node $x \in (A \setminus \{00^n\}) \cup B$. Since the triangulation is symmetric on the boundary with respect to θ_k, we can immediately also extend this association to the outer boundary of $R(i, i+1)$ for all other i. In other words, for any $i \in \mathbb{Z}_k$ and any $x \in (A \setminus \{00^n\}) \cup B$, we let $K_i(x) = (\theta_k)^{i-1}(K_1(x))$, which is an interval of $4k$ vertices on $\mathsf{conv}(\{\mathbf{u}_i, \mathbf{u}_{i+1}\})$. Thus, for any x there are k distinct intervals on the boundary associated to it, one in each of $\mathsf{conv}(\{\mathbf{u}_i, \mathbf{u}_{i+1}\})$, $i \in \mathbb{Z}_k$.

In the rest of this proof we explain how to assign a label in $\mathbb{Z}_k$ to every vertex of the triangulation so that the k-polygon-Tucker boundary conditions are satisfied and any solution (i.e., a trichromatic triangle or an edge with non-consecutive labels) must contain a vertex that lies in some interval $K_i(x)$ where x is a solution-node of Bipartite-mod-k[#1] (i.e., a node with degree $\neq 0 \mod k$). This will ensure that from any solution of the k-polygon-Tucker instance we can easily obtain a solution-node of the Bipartite-mod-k[#1] instance.

We begin by defining the "environment" label for every vertex of the triangulation. This corresponds to the standard label that the vertex will have, unless we specify it otherwise in the construction. Any vertex lying in $R(i, i+1) \setminus \mathsf{conv}(\{\mathbf{0}, \mathbf{u}_{i+1}\})$ has the environment label i. Next, we define "cables", which will be used to embed the edges of the Bipartite-mod-k[#1] instance in our construction.

Wires and Cables. A "wire" has an associated label $i \in \mathbb{Z}_k$ and it simply consists of a path of vertices in the triangulation such that all vertices on the path have the label i. A "cable" is made out of $k-1$ wires, where each wire has a distinct associated label. The wires are arranged in parallel inside the cable so that only wires with consecutive labels are allowed to touch. More precisely, if the cable uses labels $\mathbb{Z}_k \setminus \{i\}$, then the wires are arranged according to their labels in the order $i+1, i+2, \dots, i+(k-1)$ from right to left, in the forward direction of the cable. Note that while wires are not directed, we can define a direction for every cable, based on the order of the wires inside the cable. A cable using the labels $\mathbb{Z}_k \setminus \{i\}$ is only allowed to exist inside a region with environment label i. This ensures that any vertex that is adjacent to either side of the cable, and thus to the wire labeled $i+1$ or $i+(k-1)$, is labeled i and does not introduce a solution. The construction of the cables ensures that the wires are "isolated" from each other and from the environment, in the sense that no solution is introduced along the cable. However, if the start or end of a cable using the labels $\mathbb{Z}_k \setminus \{i\}$ is allowed to "touch" the environment label i, then this will necessarily yield a trichromatic triangle at that point. See Figure 3 for an illustration of how a cable is constructed.

It is easy to see that a cable can turn without introducing solutions. Next, let us see how a cable can transition from one environment to another. Consider a cable in environment $i \in \mathbb{Z}_k$, i.e., it uses the labels $\mathbb{Z}_k \setminus \{i\}$ and lies in $R(i, i+1)$. If the cable arrives on the boundary $\mathsf{conv}(\{\mathbf{0}, \mathbf{u}_{i+1}\})$ of $R(i, i+1)$, we can transform it into a cable that uses the labels $\mathbb{Z}_k \setminus \{i+1\}$ and continues into $R(i+1, i+2)$, i.e., in the environment $i+1$, on the other side of $\mathsf{conv}(\{\mathbf{0}, \mathbf{u}_{i+1}\})$. Importantly, the direction of the cable does not change and we do not introduce any new solutions. This transformation of the cable is shown in Figure 4. The idea is simple. Consider a cable in environment i that arrives on $\mathsf{conv}(\{\mathbf{0}, \mathbf{u}_{i+1}\})$ moving forward. The wires are arranged according to their labels in the order $i+1, i+2, \dots, i+(k-1)$ from right to left, in the forward direction of

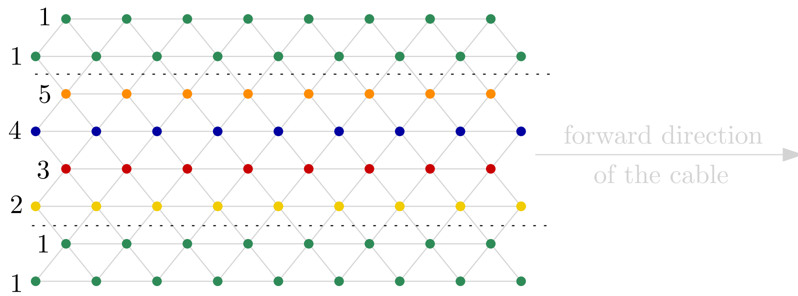

Figure 3: A cable for the case $k = 5$. The cable uses the labels $\mathbb{Z}_5 \setminus \{1\}$ and the environment has label 1. The labels are color-coded as indicated on the left-hand side, i.e., green is label 1, yellow is label 2 etc. The forward direction of the cable is indicated by an arrow on the right-hand side. Note that the portion of the cable shown in the figure does not introduce any solution of 5-POLYGON-TUCKER.

the cable. When the cable reaches $\mathsf{conv}(\{\mathbf{0}, \mathbf{u}_{i+1}\})$, the right-most wire (which is labeled $i+1$) is dropped from the cable, i.e., it merges into the environment $i+1$ of $R(i+1, i+2)$. On the other side of the cable, a new wire with label $i = i + 1 + (k-1)$ is created by using the environment i of $R(i, i+1)$. Thus, we obtain a cable with wires $i+2, \ldots, i+1+(k-1)$ (from right to left) in the environment $i+1$, as desired. It is easy to see that this construction does not introduce any new solutions, because we have ensured that non-consecutive labels do not "touch".

When a cable starts or ends on the outer boundary of $R(i, i+1)$, i.e., on $\mathsf{conv}(\{\mathbf{u}_i, \mathbf{u}_{i+1}\})$, the k-POLYGON-TUCKER boundary conditions force a cable to start or end at the corresponding position in each of the regions $R(j, j+1)$, $j \in \mathbb{Z}_k \setminus \{i\}$. These $k-1$ cables will have the same direction as the original cable. In other words, if the cable in region $R(i, i+1)$ ends on the outer boundary, then the $k-1$ corresponding cables will also end on their corresponding boundary. Similarly, if the original cable starts on the boundary, then the $k-1$ corresponding cables will start on their corresponding boundary.

Construction of the instance. Before we begin, let us introduce the following useful terminology. Every node x of the BIPARTITE-MOD-k[#1] instance has some number $\ell \in \{0, 1, \ldots, k\}$ of neighbors, as given by $C(x)$. For any $i \in [\ell]$, we define the "ith neighbor of x" to be the ith node in the lexicographically ordered list of neighbors of x.

We are now ready to begin describing the instance we construct. Recall that we have defined an environment label for every vertex of the triangulation except $\mathbf{0}$. Now consider the labeling where every vertex is simply labeled by its environment label. Clearly, this labeling satisfies the k-POLYGON-TUCKER boundary conditions. Furthermore, no matter how we pick the label of $\mathbf{0}$, there will be a solution there, since $\mathbf{0}$ is adjacent to all environments. Now it is easy to see that we can "move" this solution by locally modifying some of the labels. Namely, instead of having all labels meet at $\mathbf{0}$, we can instead construct a cable that uses the labels $\mathbb{Z}_k \setminus \{1\}$ and moves into the region with environment label 1. As a result, there is no longer a solution at $\mathbf{0}$, but instead there is now a solution at the end of the cable. In more detail, every environment label except 1 yields a wire with the corresponding label and the wires are arranged into a cable that uses the labels $\mathbb{Z}_k \setminus \{1\}$. Figure 5 illustrates this construction in the case $k = 5$.

Recall that the node $00^n \in A$ has exactly one neighbor $y \in B$. Let $j \in [k]$ be the number such that 00^n is the jth neighbor of y. The cable that we just constructed at the center of the instance

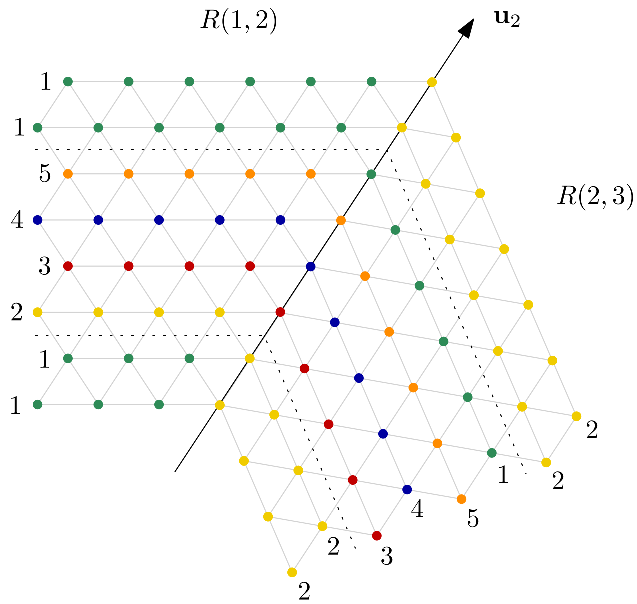

Figure 4: Transformation of a cable for the case $k = 5$. In the region $R(1,2)$ the cable uses labels $\mathbb{Z}_5 \setminus \{1\}$ and has environment 1. When the cable reaches region $R(2,3)$, the construction shown in the figure ensures that from now on, the cable uses labels $\mathbb{Z}_5 \setminus \{2\}$ and has environment 2. The cable uses the labels $\mathbb{Z}_5 \setminus \{2\}$ and the environment has label 2. Note that the transformation of the cable as shown in the figure does not introduce any solution of 5-POLYGON-TUCKER.

will be routed so that it ends at $K_j(y)$ (a segment on the outer boundary of $R(j, j+1)$, as defined above). Furthermore, for any $x \in A \setminus \{00^n\}$ and $y \in B$ such that x and y are neighbors, there will be a cable starting at $K_i(x)$ and ending at $K_j(y)$, where $i, j \in [k]$ are such that x is the jth neighbor of y, and y is the ith neighbor of x. This ensures that the following properties hold.

Consider a node $x \in A \setminus \{00^n\}$ that has ℓ neighbors, where $\ell \in \{0, 1, \dots, k\}$. If $\ell = 0$, i.e., x is an isolated node, then for all $i \in [k]$ there is no cable at $K_i(x)$. In particular, there is no k-POLYGON-TUCKER solution in any $K_i(x)$. If $\ell \in [k]$, then for all $i \in [\ell]$ there is a cable starting at $K_i(x)$ (and ending at some $K_j(y)$). For all $i \in [k] \setminus [\ell]$, there is a start of a cable at $K_i(x)$, but the cable just stops immediately and does not go anywhere. This ensures that the k-POLYGON-TUCKER boundary conditions are satisfied, but also that there is a k-POLYGON-TUCKER solution at $K_i(x)$ for all $i \in [k] \setminus [\ell]$ (because of an exposed end of cable). Thus, we obtain that

- if $\ell \in [k-1]$, there is a solution at $K_i(x)$ for some $i \in [k]$,
- if $\ell \in \{0, k\}$, then there is no solution in any $K_i(x)$.

Similarly, consider a node $y \in B$ that has ℓ neighbors, where $\ell \in \{0, 1, \dots, k\}$. If $\ell = 0$, i.e., y is an isolated node, then for all $j \in [k]$ there is no cable at $K_j(y)$. In particular, there is no k-POLYGON-TUCKER solution in any $K_j(y)$. If $\ell \in [k]$, then for all $j \in [\ell]$ there is a cable ending at $K_j(y)$ (that started at some $K_i(x)$). For all $j \in [k] \setminus [\ell]$, there is an end of a cable at $K_j(y)$, but the cable just started there and does not come from anywhere else. This ensures that the k-POLYGON-TUCKER boundary conditions are satisfied, but also that there is a k-POLYGON-TUCKER solution at $K_j(y)$ for all $j \in [k] \setminus [\ell]$ (because of an exposed start of cable). Thus, we obtain that

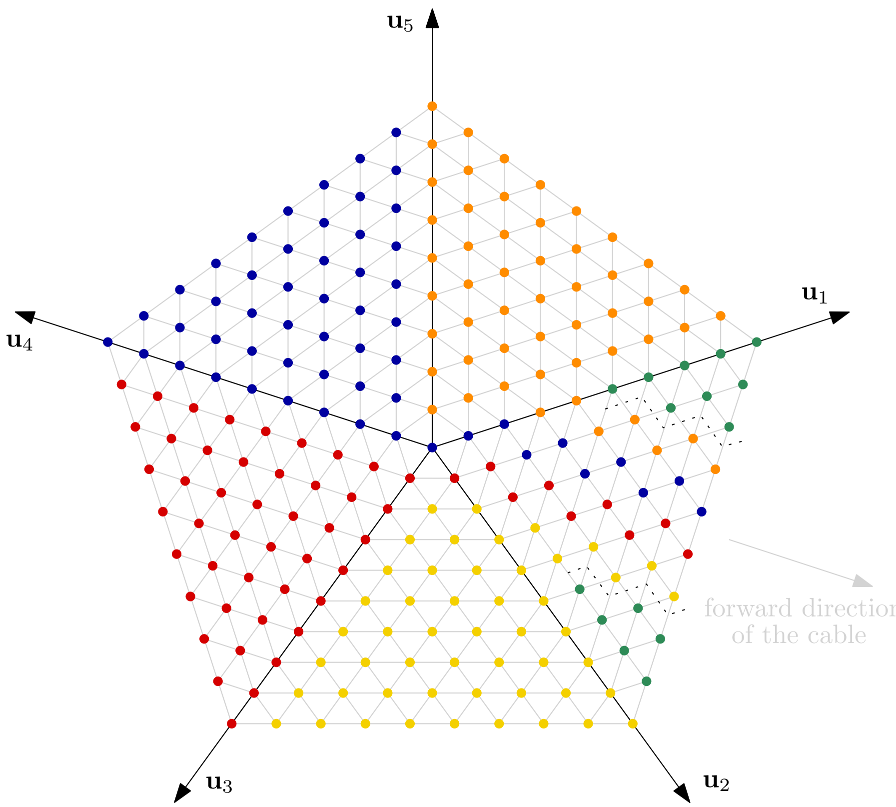

Figure 5: Construction around the origin for the case $k = 5$. Even though all the environments "meet" at the origin, the construction shown in the figure ensures that there is no solution around the origin, but instead a cable is created. The labels are color-coded as in Figure 3 and Figure 4.

- if $\ell \in [k-1]$, there is a solution at $K_j(y)$ for some $j \in [k]$,
- if $\ell \in \{0, k\}$, then there is no solution in any $K_j(y)$.

As a result, if the cables can indeed be constructed to connect the various $K_i(x)$ and $K_j(y)$ as desired, then any k-POLYGON-TUCKER solution of the instance will have to be next to some $K_i(x)$ such that x is a solution-node or next to some $K_j(y)$ such that y is a solution node. One immediate obstacle to routing the cables as desired is that we are working in two dimensions and it is very likely that cables will have to cross each other. Fortunately, there is a simple "trick" that has been used in prior work to resolve this issue [Chen and Deng, 2009]. Consider the following idea: cut the two cables that want to cross each other at the point of crossing. This creates two ends of cables and two starts of cables. It is easy to see that we can connect an end of cable with a start of cable, and the other end with the other start, so that no crossing occurs anymore. This modification of the cables is completely local and does not have any impact on the rest of the instance.

Constructing the labeling function. Since we want to be able to construct a circuit for the labeling function, we need to be a bit more precise about the path followed by every cable. One way to achieve this is to reserve a separate "circular lane" for each pair (y, j), where $y \in B$ and $j \in [k]$. A circular lane is a path of sufficient width that simply stays parallel to the outer boundary in each region $R(i, i+1)$ and thus makes a full "circle" around the center of the domain. By picking a fine enough triangulation, we can ensure that there is a separate, disjoint circular lane $L_{y,j}$ for each pair (y, j), where $y \in B$ and $j \in [k]$. Then the cable going from some $K_i(x)$ to some $K_j(y)$ will be routed as follows. Starting at $K_i(x)$, move perpendicularly to the outer boundary towards the inside of the domain, until the circular lane $L_{y,j}$ is reached. Next, follow the lane in clockwise direction. When following the lane, the cable might have to transition from some environment to the next and this is implemented as described earlier. The cable stops following the lane when it reaches the part of the lane lying just "above" $K_j(y)$. In other words, the cable stops following the lane when it is at the point where it can just turn left and move straight towards the boundary to end up at $K_j(y)$, as desired. Clearly, we can pick the triangulation fine enough so that this routing is indeed well defined.

This construction has two advantages. First of all, it ensures that any crossing involves at most two cables, not more. We can then use the trick described above to locally resolve these crossings. Furthermore, it ensures that we can construct a Boolean circuit computing the labeling function. Indeed, given any vertex of the triangulation we can easily determine on which circular lane and on which path perpendicular to the outer boundary it lies. This gives us enough information to then use the BIPARTITE-MOD-k[#1] circuit C as a sub-routine to determine whether the vertex lies on a cable and if so, how exactly the cable behaves locally (including possible crossing-avoiding trick). The circuit C only needs to be queried a constant number of times and thus the resulting circuit for the labeling will have polynomial size with respect to the size of C and n. We omit the full details, since this part of the proof is essentially the same as in prior work (see e.g., [Chen and Deng, 2009]). ☐

Based on the results presented in this and the previous section we get the following theorem.

Theorem 3.16. *The problems k-POLYGON-TUCKER and k-POLYGON-BORSUK-ULAM are both* PPA-k[#1]*-complete.*

In particular, if $k = p^r$ is a prime power, then k-POLYGON-TUCKER and k-POLYGON-BORSUK-ULAM are PPA-p-complete.

4 The BSS Theorem is PPA-p-complete

The main result of this section is the PPA-p-completeness of the computational problem associated with the BSS Theorem. We refer to this problem as p-BSS and we define it formally in Section 4.3. We state the main theorem of the section below.

Theorem 4.1. *For every prime p, the problem p-*BSS *is* PPA-p*-complete.*

In order to prove the theorem, first we show that the BSS theorem is equivalent to a generalization of Tucker's Lemma, which we call BSS-TUCKER, and then we define the corresponding computational problems and show that they are equivalent. Then, we show the PPA-p-completeness of BSS-TUCKER, which then implies Theorem 4.1. We show the PPA-p-hardness via a reduction from the p-POLYGON-TUCKER problem, proven to be PPA-p-complete in the previous section. The membership in PPA-p is proven in Section 5, where we show that BSS-TUCKER reduces to another variant of Tucker's lemma, which we call $\mathbb{Z}_p$-STAR-TUCKER (Proposition 5.4), and for which we prove membership in PPA-p.

4.1 The BSS Theorem and Equivalent Formulations

Our notation follows that of Bárány et al. [1981]. Let $p \geq 2$ prime, $n \in \mathbb{N}$ and let $P = \{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p) \in (\mathbb{R}^n)^p : \sum_{i=1}^{p} \mathbf{v}_i = \mathbf{0}\}$ and $P_2 = \{\mathbf{u} \in P : \|\mathbf{u}\|_2 \leq 1\}$. It holds that

$$P \cong \mathbb{R}^{n(p-1)} \text{ and } P_2 \cong B^{n(p-1)}.$$

Note that P is a hyperplane of $\mathbb{R}^{np}$ with dimension $n(p-1)$. Let $\mathcal{B}$ be an orthogonal basis of P in $\mathbb{R}^{np}$ and let $\phi : P_2 \to B^{n(p-1)}$ be the function that maps $\mathbf{x} \in P_2$ to its coefficients in base $\mathcal{B}$. Then, ϕ is a homeomorphism of P_2 and $B^{n(p-1)}$. Notice that $\phi(\partial P_2) = S^{n(p-1)-1}$ and that ϕ and ϕ^{-1} are efficiently computable. The same mapping shows that P and $\mathbb{R}^{n(p-1)}$ are homeomorphic.

Let

$$\theta(\mathbf{v}_1, \ldots, \mathbf{v}_p) = (\mathbf{v}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{p-2}, \mathbf{v}_{p-1}).$$

The function θ has order p; namely the composition by itself p times is equal to the identity. Also, for all $i \in \{1, \ldots, p-1\}$ and all $\mathbf{x} \in P \setminus \{\mathbf{0}\}$, $\theta^i(\mathbf{x}) \neq \mathbf{x}$. In other words, for any $i < p$, θ^i restricted to $P \setminus \{\mathbf{0}\}$ has no fixed points. Hence, θ acts freely on $P \setminus \{\mathbf{0}\}$. It follows with a similar argument that the function θ acts freely also on $P_2 \setminus \{\mathbf{0}\}$ and on ∂P_2.

The original statement of the BSS Theorem requires the notion of CW-complexes, but since in our work we do not use CW-complexes, we will not define them formally. Intuitively, a CW-complex consists of building blocks that can be topologically glued together.

Let p be a prime, $n \geq 1$ and X be a CW-complex consisting of p copies of the $n(p-1)$-dimensional ball glued on their boundaries.

Let $\alpha : \mathbb{R}^{n(p-1)} \to \mathbb{R}^{n(p-1)}$ such that $\alpha = \phi \circ \theta \circ \phi^{-1}$. Note that α acts freely on $\mathbb{R}^{n(p-1)} \setminus \{\mathbf{0}\}$, $B^{n(p-1)} \setminus \{\mathbf{0}\}$ and $S^{n(p-1)-1}$. Let ω be the extension of α on X defined as follows:

$$\omega(\mathbf{y}, r, q) = (\alpha \mathbf{y}, r, q+1 \pmod{p}),$$

where $(\mathbf{y}, r, q)$ denotes the point of the q-th ball with radius r and direction $\mathbf{y} \in S^{n(p-1)-1}$.

The map ω is a free action on X and the following theorem holds:

Theorem 4.2 (BSS Theorem, [Bárány, Shlosman, and Szücs, 1981])**.** *For the mapping ω and any continuous map $h : X \to \mathbb{R}^n$, there exists an $\mathbf{x} \in X$ such that $h(\mathbf{x}) = h(\omega \mathbf{x}) = \cdots = h(\omega^{p-1} \mathbf{x})$.*

The following equivalent formulations of Theorem 4.2 are useful for defining the computational problem related to BSS.

Theorem 4.3 (BSS Theorem, equivalent formulations)**.** *The following statements are equivalent to the BSS Theorem:*

1. *Let $g : P_2 \to P$ be continuous and such that $g(\theta \mathbf{x}) = \theta g(\mathbf{x})$ for all $\mathbf{x} \in \partial P_2$. Then, there exists $\mathbf{x} \in P_2$ such that $g(\mathbf{x}) = 0$.*

2. *Let $g : B^{n(p-1)} \to \mathbb{R}^{n(p-1)}$ be continuous and such that $g(\alpha \mathbf{x}) = \alpha g(\mathbf{x})$ for all $\mathbf{x} \in S^{n(p-1)-1}$. Then, there exists $\mathbf{x} \in B^{n(p-1)}$ such that $g(\mathbf{x}) = 0$.*

Proof. We first show that the two statements are equivalent and then that statement (2) is equivalent to Theorem 4.2.

(1) $\Longleftrightarrow$ (2) The equivalence follows from $P_2 \cong B^{n(p-1)}$ and $P \cong \mathbb{R}^{n(p-1)}$, as well as from the equivariance of ϕ.

(2) $\Longleftrightarrow$ (BSS) We first show that the BSS Theorem implies (2). Recall that the CW-complex X consists of p copies of $B^{n(p-1)}$ with their boundaries "glued" together. Define $h : X \to \mathbb{R}^n$ as follows: for $\mathbf{x} = (\mathbf{y}, r, i) \in X$, let $h(\mathbf{x}) = [\phi^{-1} \circ g \circ \alpha^{1-i}(r\mathbf{y})]_{2-i} \in \mathbb{R}^n$, where for $j \in \mathbb{Z}_p \equiv [p]$, $[\cdot]_j$ denotes the j-th component of an element in $(\mathbb{R}^n)^p$ (i.e., $[(\mathbf{v}_1, \dots, \mathbf{v}_p)]_j = \mathbf{v}_j$). The mapping h is well-defined and continuous on the glued boundary, because for all i and $\mathbf{y} \in S^{n(p-1)-1}$, we have $h(\mathbf{y}, 1, i) = [\phi^{-1} \circ g \circ \alpha^{1-i}(\mathbf{y})]_{2-i} = [\theta^{1-i}\phi^{-1} \circ g(\mathbf{y})]_{2-i} = [\phi^{-1} \circ g(\mathbf{y})]_1$ (which does not depend on i). Finally, note that any $\mathbf{x} = (\mathbf{y}, r, 1) \in X$ with $h(\mathbf{x}) = h(\omega\mathbf{x}) = \cdots = h(\omega^{p-1}\mathbf{x})$ yields $\mathbf{z} = r\mathbf{y} \in B^{n(p-1)}$ with $[\phi^{-1} \circ g(\mathbf{z})]_1 = \cdots = [\phi^{-1} \circ g(\mathbf{z})]_p$, which implies $\phi^{-1} \circ g(\mathbf{z}) = 0$ (by definition of P), and thus $g(\mathbf{z}) = 0$.

Conversely, (2) implies BSS. We identify $B^{n(p-1)}$ with the first ball in the CW-complex X. Given a continuous function $h : X \to \mathbb{R}^n$, define $g : B^{n(p-1)} \to \mathbb{R}^{n(p-1)}$ by $g(\mathbf{x}) = \phi(h(\omega^p\mathbf{x}) - h(\omega^{p-1}\mathbf{x}), h(\omega^{p-1}\mathbf{x}) - h(\omega^{p-2}\mathbf{x}), \dots, h(\omega\mathbf{x}) - h(\mathbf{x}))$. It is easy to check that $g(\alpha\mathbf{x}) = \alpha g(\mathbf{x})$ for all $\mathbf{x} \in S^{n(p-1)-1}$ by noting that $\omega\mathbf{x} = \alpha\mathbf{x}$ for such $\mathbf{x}$. □

4.2 The BSS-Tucker Lemma

In this section, we define a generalization of Tucker's Lemma, that we call the BSS-Tucker Lemma and show that it is equivalent to the BSS Theorem. The BSS-Tucker Lemma applies to triangulations of P_2 that have some special properties.

For $j \in [n]$, let $\mathbf{e}_j \in \mathbb{R}^n$ be the j-th unit vector. For $(i, j) \in \mathbb{Z}_p \times [n]$, let

$$\mathbf{e}^{i,j} = \frac{1}{2(p-1)}(-\mathbf{e}_j, \dots, -\mathbf{e}_j, (p-1)\mathbf{e}_j, -\mathbf{e}_j, \dots, -\mathbf{e}_j) \in P$$

where the term $(p-1)\mathbf{e}_j$ is in the i-th position. For each $\mathbf{s} = (s_1, \dots, s_n) \in [p]^n$, we define the simplex $\sigma_{\mathbf{s}} = \{\mathbf{0}\} \cup \{\mathbf{e}^{i,j}$ s.t. for each $j \in [n], i \in \mathbb{Z}_p \setminus \{s_j\}\}$. Note that $|V(\sigma_{\mathbf{s}})| = (p-1)n + 1$.

Lemma 4.4. *$T^* = \{\tau : \tau \subseteq \sigma_{\mathbf{s}}$ with $\mathbf{s} \in [p]^n\}$ is a triangulation of P_2.*

Proof. Let $C = \mathrm{conv}\left((\mathbf{e}^{i,j})_{(i,j)\in\mathbb{Z}_p\times[n]}\right)$. Then, by definition of T^*, $C \cong \|T^*\|$. So, the lemma follows from the fact that $C \cong P_2$. □

Definition 9. We say that a triangulation T of P_2 is *nice* if it satisfies the following two conditions:

- If $\sigma \in T \cap \partial P_2$ then $\theta\sigma \in T$.
- T refines the triangulation T^* (that is, for each $\sigma \in T$ there is $\tau \in T^*$ with $\sigma \subseteq \tau$).

Theorem 4.5 (BSS-Tucker Lemma)**.** *Let T be a nice triangulation of P_2. Let $\lambda = (\lambda_1, \lambda_2) : V(T) \to \mathbb{Z}_p \times [n]$ be a labeling such that for all $\mathbf{x} \in \partial T$, $\lambda(\theta\mathbf{x}) = (\lambda_1(\mathbf{x}) + 1, \lambda_2(\mathbf{x}))$. Then, there exists a $(p-1)$-simplex σ of T such that $\lambda(\sigma) = \mathbb{Z}_p \times \{j\}$ for some $j \in [n]$, where $\lambda(\sigma) = \{\lambda(\mathbf{x}) : \mathbf{x} \in V(\sigma)\}$.*

Lemma 4.6. *The BSS Theorem (Theorem 4.3) implies the* BSS-Tucker *Lemma (Theorem 4.5).*

Proof. We interpret each label $(i, j) \in \mathbb{Z}_p \times [n]$ as the vector $\mathbf{e}^{i,j}$ and we set g to be the extension of λ to a piecewise linear function on P_2. Notice that $g(\theta\mathbf{x}) = \theta g(\mathbf{x})$ for $\mathbf{x} \in \partial P_2$. Then, it follows from Theorem 4.3 that there exists an $\mathbf{x} \in P_2$ such that $g(\mathbf{x}) = 0$. The lemma follows by noting that any convex combination of different vectors $\mathbf{e}^{i,j}$ that equals $\mathbf{0}$ must contain p vectors $\mathbf{e}^{i_1,j_1}, \dots, \mathbf{e}^{i_p,j_p}$ such that $\{i_k, j_k\}_{k\in[p]} = \mathbb{Z}_p \times \{j\}$. Hence, the point $\mathbf{x}$ lies in a simplex with a $(p-1)$-dimensional face σ such that $\lambda(\sigma) = \mathbb{Z}_p \times \{j\}$ for some $j \in [n]$. □

Lemma 4.7. *The* BSS-Tucker *Lemma (Theorem 4.5) implies the BSS Theorem (Theorem 4.3).*

Proof. We show that BSS-Tucker implies Statement (1) of Theorem 4.3. Using standard arguments, it suffices to show that for every $\varepsilon > 0$ we can find a point $\mathbf{x}$ such that $\|g(\mathbf{x})\|_\infty \leq \varepsilon$. Since $g : P_2 \to P$ is a continuous function in a compact set, it is also uniformly continuous. Thus, for every $\varepsilon > 0$, there exists δ such that if $\|\mathbf{x} - \mathbf{x}'\|_2 < \delta$, then $\|g(\mathbf{x}) - g(\mathbf{x}')\|_\infty < \varepsilon/n$. Assume that T is a nice triangulation of P_2 with diameter at most δ.

For any point $\mathbf{x} \in V(T)$, the label $\lambda(\mathbf{x}) = (i^*, j^*)$ is defined as follows. For $(i,j) \in \mathbb{Z} \times [n]$, let $[g(\mathbf{x})]_{i,j}$ denote the (i,j)-coordinate of $g(\mathbf{x}) \in P$. First, consider the case where $g(\mathbf{x}) \neq \mathbf{0}$. Then, pick $j^* = \operatorname{argmax}_{j\in[n]} \max_{i\in[p]}[g(\mathbf{x})]_{i,j}$. Break ties by picking the smallest such j. Let $S = \{i : [g(\mathbf{x})]_{i,j^*} = \max_\ell [g(\mathbf{x})]_{\ell,j^*}\}$ and pick $i^* = T_p(S)$, where T_p is the $\mathbb{Z}_p$-equivariant tie-breaking function of Definition 19. Note that $S \notin \{\emptyset, [p]\}$, because $g(\mathbf{x}) \neq \mathbf{0}$ and by the choice of j^*. If $g(\mathbf{x}) = \mathbf{0}$, then if $\mathbf{x} = \mathbf{0}$, we assign an arbitrary label, and if $\mathbf{x} \neq \mathbf{0}$, we use the same procedure as above but with $\mathbf{x}$ instead of $g(\mathbf{x})$.

With this definition of the labeling, it is easy to check that for any $\mathbf{x} \in \partial T$, we always have $\lambda(\theta\mathbf{x}) = (\lambda_1(\mathbf{x}) + 1, \lambda_2(x))$, since $g(\theta\mathbf{x}) = \theta g(\mathbf{x})$. Hence, by Theorem 4.5 there exists a $\sigma = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \in T$ and a $j^* \in [n]$ such that $\lambda(\mathbf{x}_i) = (i, j^*)$ for all $i \in [p]$.

The lemma follows by showing that there exists $i \in [p]$ such that $\|g(\mathbf{x}_i)\|_\infty \leq \varepsilon$. First of all, note that for any $\mathbf{x} \in P$, by definition we have that $\|g(\mathbf{x})\|_\infty \leq n \cdot \max_{i,j}[g(\mathbf{x})]_{i,j}$. Now, assume for the sake of contradiction that $\|g(\mathbf{x}_i)\|_\infty > \varepsilon$ for all $i \in [p]$. Then, it follows that $[g(\mathbf{x}_i)]_{i,j^*} > \varepsilon/n$ for all $i \in [p]$. But by the choice of diameter for the triangulation and uniform continuity of g, this implies that $[g(\mathbf{x}_1)]_{i,j^*} > 0$ for all $i \in [p]$, which contradicts $\mathbf{x}_1 \in P$. $\square$

4.3 Computational problems: p-BSS-Tucker and p-BSS

Motivated by Theorem 4.3 and Theorem 4.5, we define the computational problems corresponding to the BSS Theorem and the BSS-Tucker Lemma. Combining the proof ideas in Lemma 4.6 and Lemma 4.7, together with some efficiency requirements of the triangulation, as per Definition 17, we show that the two computational problems are polynomially equivalent.

As before, we assume that the input functions are represented as arithmetic circuits with operations $\times\zeta$, $+$, $-$, $<$, min, and max and rational constants. Similarly to our definition of k-polygon-Borsuk-Ulam, and for the same reasons, we will add a solution type to ensure that it is Lipschitz-continuous.

The computational analogue of the BSS theorem is based on statement (2) of Theorem 4.3. It takes as input an arithmetic circuit, which evaluates the function $g : B^{n(p-1)} \to \mathbb{R}^{n(p-1)}$. In order to deal with the polynomial-size representation of any of the solutions we allow as a solution any approximate violation of the equivariance condition of the BSS theorem, as we did with p-polygon-Borsuk-Ulam in Section 3.

p**-BSS:**

Input: An integer $n \geq 1$, an accuracy parameter $\varepsilon > 0$, a Lipschitz constant L, and an arithmetic circuit $\mathcal{C}$.

Output:

1. A point $\mathbf{x} \in S^{n(p-1)-1}$ such that $\|\mathcal{C}(\alpha\mathbf{x}) - \alpha\mathcal{C}(\mathbf{x})\| > \eta(\varepsilon, p) := \varepsilon/8p^4$
2. Two points $\mathbf{x}, \mathbf{y} \in B^{n(p-1)}$ such that $\|\mathcal{C}(\mathbf{x}) - \mathcal{C}(\mathbf{y})\| > L\,\|\mathbf{x} - \mathbf{y}\|$
3. A point $\mathbf{x}^* \in B^{n(p-1)}$ such that $\|\mathcal{C}(\mathbf{x}^*)\|_\infty \leq \varepsilon$

A valid output of the p-BSS problem is either a point that violates the boundary condition of Theorem 4.3, two points that violate the Lipschitz-continuity or an approximate root of the function g.

The computational analogue of the BSS-Tucker Lemma is based on Theorem 4.5 and is parameterized by a "triangulation scheme" $\mathcal{T}$. Namely, given an $m \in \mathbb{N}$, $\mathcal{T}(m)$ yields a nice triangulation T with diameter at most $1/2^m$. The triangulation T is given through two arithmetic circuits, index and value (see Definition 17), that have size polynomial in n and m. Assume that for $m = 0$, $\mathcal{T}$ yields the index^* and value^* circuits of T^*.

p**-BSS-Tucker[**$\mathcal{T}$**]:**

Input: An arithmetic circuit λ that outputs a number in $\mathbb{Z}_p \times [n]$.

Output:

1. A vertex $\mathbf{x} \in \partial T$ such that $\lambda(\theta\mathbf{x}) \neq (\lambda_1(\mathbf{x}) + 1, \lambda_2(\mathbf{x}))$
2. A simplex $\sigma^* \in T$ such that $\lambda(\sigma^*) = \mathbb{Z}_p \times \{j\}$ for some $j \in [n]$

The valid outputs of the p-BSS-Tucker problem correspond either to points that violate the boundary condition of Theorem 4.5 or to a fully labeled simplex in T.

Remark: To efficiently check whether $\mathbf{x} \in \partial T$, we use the index^* and value^* circuits of T^*, which exist by assumption on $\mathcal{T}$ for $m = 0$. If $\mathsf{value}^*(\mathsf{index}^*(\mathbf{x}))$ does not contain $\mathbf{0}$, then by definition of T^* and the fact that T is a nice triangulation $\mathbf{x} \in \partial T$.

Theorem 4.8. *p-BSS-Tucker[$\mathcal{T}$] and p-BSS are polynomially equivalent.*

Proof. First, we show that p-BSS-Tucker[$\mathcal{T}$] reduces to p-BSS and then that p-BSS reduces to p-BSS-Tucker[$\mathcal{T}$]. For simplicity of the presentation and in order for the main ideas to be clear we present here a proof sketch of these reductions where we assume that $\eta(\varepsilon, p) = 0$ in the p-BSSproblem. The complete proof then follows using the tedious but straightforward case analysis that we used in the proof of Lemma 3.13.

p**-BSS-Tucker[**$\mathcal{T}$**]** $\leq$ p**-BSS:** Pick $\varepsilon < \frac{1}{(np)^2}$. Define the circuit $\mathcal{C}$ using the procedure described in Lemma 4.6 and the homeomorphism ϕ of $B^{n(p-1)}$ and P_2. Note that $\mathcal{C}$ is L-Lipschitz-continuous for some $L = O(2^m)$. Thus, a solution of this instance of p-BSS is:

1. a point $\mathbf{x} \in S^{n(p-1)-1}$ such that $\mathcal{C}(\alpha\mathbf{x}) \neq \alpha\mathcal{C}(\mathbf{x})$. In this case, let σ be the simplex such that $\phi^{-1}(\mathbf{x}) \in \|\sigma\|$, then there exists a vertex $\mathbf{v} \in V(\sigma)$ such that $\lambda(\theta\mathbf{v}) \neq (\lambda_1(\mathbf{v}) + 1, \lambda_2(\mathbf{v}))$. Observe that $\sigma = \mathsf{value}(\mathsf{index}(\phi^{-1}(\mathbf{x})))$ and that it has at most $n(p-1)+1$ vertices. Hence, we can efficiently find $\mathbf{v}$.

2. a point $\mathbf{x}^* \in B^{n(p-1)}$ such that $\|\mathcal{C}(\mathbf{x}^*)\|_\infty \le \varepsilon$. In this case, $\phi^{-1}(\mathbf{x}^*)$ must lie in a simplex with a fully labeled $(p-1)$-dimensional face; this follows from the choice of ε and the proof ideas of Lemma 4.6. Observe that $\phi^{-1}(\mathcal{C}(\mathbf{x}^*))$ is the convex combination of at most $n(p-1)+1$ different $e^{i,j}$'s. Hence, there must be a vector $\mathbf{e}^{i^*,j^*}$ that appears in the convex combination with coefficient at least $\frac{1}{n(p-1)+1}$. If there is an i' such that $\mathbf{e}^{i',j^*}$ does not appear in the convex combination, then $\phi^{-1}(\mathcal{C}(\mathbf{x}^*))$ has at least one coordinate at least as large as $\frac{1}{n(p-1)+1}$. Hence, $\|\phi^{-1}(\mathcal{C}(\mathbf{x}^*))\|_2 \ge \|\phi^{-1}(\mathcal{C}(\mathbf{x}^*))\|_\infty \ge \frac{1}{n(p-1)+1}$, which means that $\|\mathcal{C}(\mathbf{x}^*)\|_2 \ge \frac{1}{n(p-1)+1}$. This is a contradiction since it implies that $\|\mathcal{C}(\mathbf{x}^*)\|_\infty \ge \frac{\|\mathcal{C}(\mathbf{x}^*)\|_2}{\sqrt{np}} \ge \frac{1}{(np)^2} > \varepsilon$.

   The point $\phi^{-1}(\mathbf{x}^*)$ lies in the simplex $\sigma = \mathsf{value}(\mathsf{index}(\phi^{-1}(\mathbf{x})))$, which has at most $n(p-1)+1$ vertices. Hence, finding the $(p-1)$-dimensional fully labeled face can be done efficiently.

p-BSS $\le$ p-BSS-Tucker[$\mathcal{T}$]: Set m in $\mathcal{T}$ such that $1/2^m \le \frac{\varepsilon}{nL}$. The labeling λ is defined as in Lemma 4.7 using as function $g : P_2 \to P$ the function given by $\phi^{-1} \circ \mathcal{C} \circ \phi$, where ϕ is the homeomorphism of $B^{n(p-1)}$ and P_2; note that all operations can be described with an arithmetic circuit. A solution of this instance of p-BSS-Tucker[$\mathcal{T}$] is:

1. a vertex $\mathbf{x} \in \partial T$ such that $\lambda(\theta\mathbf{x}) \ne (\lambda_1(\mathbf{x})+1, \lambda_2(\mathbf{x}))$. In this case, it must hold that $\mathcal{C}(\alpha \circ \phi(\mathbf{x})) \ne \alpha\mathcal{C}(\phi(\mathbf{x}))$. Thus, $\phi(\mathbf{x})$ is a solution of p-BSS.
2. a simplex $\sigma^* \in T$ such that $\lambda(\sigma^*) = \mathbb{Z}_p \times \{j\}$ for some $j \in [n]$. In this case, following the proof of Lemma 4.7, there exists a vertex $\mathbf{x}$ in σ^* such that $\|g(\mathbf{x})\|_\infty \le \varepsilon$ (or a violation of L-Lipschitz-continuity). Then, since ϕ as defined in Section 4.1 preserves the ℓ_2 distances between P_2 and $B^{n(p-1)}$, $\|\mathcal{C}(\phi(\mathbf{x}))\|_\infty \le \|\mathcal{C}(\phi(\mathbf{x}))\|_2 \le \varepsilon$. Thus, $\phi(\mathbf{x})$ is a solution of p-BSS.

□

Remark 4 (Kuhn's triangulation for p-BSS-Tucker). In order to use Kuhn's triangulation we work on the domain $C_\infty = \{(c^1,\dots,c^p) \in ([0,1]^n)^p | \forall j \in [n], \exists i \in [p] : c^i_j = 0\}$ instead of P_2. These are coordinates with respect to the vectors $e^{i,j}$. Note that $C_\infty \cong P_\infty$, where $P_\infty = \{\sum_{i,j} c^i_j e^{i,j} | (c^1,\dots,c^p) \in C_\infty\}$, and clearly $P_\infty \cong P_2$.

We can triangulate C_∞ by using Kuhn's triangulation (Definition 18) to triangulate each cube $\{(c^1,\dots,c^p) \in C_\infty | \forall j \in [n] : c^{i_j}_j = 0\}$ for each $(i_1,\dots,i_n) \in [p]^n$. By the properties of Kuhn's triangulation, it immediately follows that this yields a triangulation of C_∞. In particular, the triangulations of two cubes "match" on their common subspace. Since we constructed the triangulation separately on each cube, it follows that it refines the triangulation T^*. Furthermore, for any simplex σ lying on the boundary of C_∞, it follows that $\theta\sigma$ is also a simplex of the triangulation. This is easy to see, because θ just changes the order of the coordinates, and the Kuhn triangulation is invariant with respect to such transformations by definition. Thus, Kuhn's triangulation is indeed a nice triangulation.

4.4 BSS-Tucker is PPA-p-complete

Having defined the computational problems corresponding to the BSS Theorem and to BSS-Tucker, we are ready to prove Theorem 4.1. The following theorem follows from Proposition 5.4, presented in Section 5.

Theorem 4.9. *For any prime p, p-BSS-Tucker[$\mathcal{T}$] is in* PPA-p.

Next we prove that p-BSS-Tucker is PPA-p-hard, through a reduction from p-polygon-Tucker.

Theorem 4.10. *For any prime $p \geq 3$, p-BSS-Tucker is PPA-p-hard, even for fixed dimension $n \geq 1$.*

Note that for $p = 2$, p-BSS-Tucker corresponds to the standard version of Tucker's lemma, which is known to be PPA-hard for any $n \geq 2$ [Aisenberg et al., 2020].

Proof. Here we prove that p-BSS-Tucker is PPA-p-hard for $n = 1$. The hardness for any $n \geq 1$ then follows from Lemma B.1, which gives a reduction from n to $n + 1$.

To show the hardness for $n = 1$ we will reduce from p-polygon-Tucker to p-BSS-Tucker. Instead of the triangulation we used in the presentation of p-polygon-Tucker, we will use Kuhn's triangulation (Definition 18). It can be shown that the hardness of p-polygon-Tucker proved in Proposition 3.15, also holds if we use Kuhn's triangulation, since there is a simple homeomorphism between the two domains. We omit the details for this.

Let λ be an instance of p-polygon-Tucker with Kuhn's triangulation of size m. Thus, the domain for this problem can be written as

$$A = \{(c_1, \ldots, c_p) \in (U_m)^p | \exists i \in [p], \forall j \notin \{i, i+1\} : c_j = 0\}.$$

We construct an instance of p-BSS-Tucker with $n = 1$ on the domain C_∞ with Kuhn's triangulation of size m. Recall that the set of vertices can be written as $\overline{C}_\infty := \{(c_1, \ldots, c_p) \in U_m^p | \exists i \in \mathbb{Z}_p : c_i = 0\}$. Let $D := \cup_{i \in \mathbb{Z}_p} D_i$, where $D_i := \{(c_1, \ldots, c_p) \in \overline{C}_\infty | \forall j \notin \{i, i+1\} : c_j = 0\}$.

We are going to embed the p-polygon-Tucker domain A into D in the most natural way. Since we are using Kuhn's triangulation, the restriction of the triangulation of $\overline{C}_\infty$ to D corresponds to Kuhn's triangulation on that domain, and thus to Kuhn's triangulation of A. We define $\lambda' : \overline{C}_\infty \to \mathbb{Z}_p$:

$$\lambda'(c_1, \ldots, c_p) = \begin{cases} \lambda(c_1, \ldots, c_p) & \text{if } (c_1, \ldots, c_p) \in D \\ T_p(\{j : c_j = 0\}) & \text{otherwise} \end{cases}$$

where T_p is the $\mathbb{Z}_p$-equivariant tie-breaking function defined in Definition 19. Note that λ' is well-defined, because if $(c_1, \ldots, c_p) \in D$, then it corresponds to a point in the p-polygon-Tucker domain. Furthermore, if $(c_1, \ldots, c_p) \notin D$, then $|\{j : c_j = 0\}| \in [1, p-1]$, so is a valid input to T_p.

Since $\theta D = D$, $\lambda(\theta c) = \lambda(c) + 1$ and $T_p(\{j : (\theta c)_j = 0\}) = T_p(\{j : c_j = 0\}) + 1$, it follows that λ' satisfies the boundary conditions.

Let $c^1, c^2, \ldots, c^p$ be a $(p-1)$-simplex of $\overline{C}_\infty$ that carries all the labels in $\mathbb{Z}_p$ (with respect to λ'). We now show how this yields a solution to the p-polygon-Tucker instance. Without loss of generality, we can assume that $c^1, c^2, \ldots, c^p$ are ordered in the order in which the simplex is defined by Kuhn's triangulation. Namely, for every $i \in \{1, \ldots, p-1\}$, there exists j_i such that $c_{j_i}^{i+1} = c_{j_i}^i + 1/m$ and $c_j^{i+1} = c_j^i$ for all $j \neq j_i$. Furthermore, the j_i are all distinct. Note that if for some i^*, $c^{i^*} \notin D$, then $c^i \notin D$ for all $i \geq i^*$. The following cases can occur:

- the simplex does not intersect D: it follows that $c^1 \neq 0 \in D$, and thus there exists j such that $c_j^1 > 0$. But then $c_j^i > 0$ for all i and the label j cannot be obtained by any vertex of this simplex.

- the intersection of the simplex with D is a face of dimension 2: then the three vertices of this face have pairwise distinct labels, and thus yield a solution to p-polygon-Tucker.

- the intersection of the simplex with D is a face of dimension 1: then it must hold that $c^1, c^2 \in D$ and $c^3, \ldots, c^p \notin D$. We distinguish between the two sub-cases:
  - $c^2_{j_1} > 0$ and $c^2_j = 0$ for all $j \neq j_1$. Then, since $c^3 \notin D$, it follows that $j_2 \notin \{j_1 - 1, j_1, j_1 + 1\}$. By definition of the j_i, we have that $c^3_{j_1} > 0$ and $c^3_{j_2} > 0$, which implies that $c^3, \ldots, c^p$ can only have labels in $\mathbb{Z}_p \setminus \{j_1, j_2\}$. As a result, c^1 and c^2 must have labels j_1 and j_2 (in any order). This yields a solution to p-POLYGON-TUCKER, because the labels are distinct and non-consecutive.
  - there exists j^* such that $c^2_{j^*} > 0$, $c^2_{j^*+1} > 0$ and $c^2_j = 0$ for all $j \notin \{j^*, j^* + 1\}$. Since $c^3 \notin D$, it must hold that $j_2 \notin \{j^*, j^* + 1\}$. Thus we have $c^3_{j_2} > 0$, and the vertices $c_3, \ldots, c_p$ can only obtain the labels $\mathbb{Z}_p \setminus \{j^*, j^* + 1, j_2\}$. It follows that the simplex cannot possibly be fully labeled.
- the intersection of the simplex with D is a face of dimension 0: then $c^2 \notin D$, and thus there exist distinct j, j' (and non-consecutive, but we don't need this here) such that $c^2_j > 0$ and $c^2_{j'} > 0$. But then, the vertices $c_2, \ldots, c_p$ can only obtain the labels $\mathbb{Z}_p \setminus \{j, j'\}$. It follows that the simplex cannot possibly be fully labeled.

This completes the proof. □

5 The $\mathbb{Z}_p$-STAR-TUCKER Lemma: Statement and PPA-p-completeness

In this section, we introduce a $\mathbb{Z}_p$-generalization of Tucker's Lemma. We further define the associated computational problem $\mathbb{Z}_p$-STAR-TUCKER, and show that it is PPA-p-complete. In the next section, we will use this problem to prove the membership of p-thief Necklace Splitting in PPA-p.

In $\mathbb{Z}_p$-STAR-TUCKER, the coordinates of the vertices lie on a star-like domain. This domain was used by Meunier [2014] for the first fully combinatorial proof of necklace splitting with p thieves.

For any prime p and any $m \geq 1$, we define $R_{p,m} = \{0\} \cup \{*^i j : i \in \mathbb{Z}_p, j \in [m]\}$, where $[m] = \{1, 2, \ldots, m\}$. For ease of notation, we also let $*^i 0 = 0$ for all $i \in \mathbb{Z}_p$. The symbols $*^1, \ldots, *^p$ should be interpreted as p different "signs" that will generalize the use of "$+$" and "$-$" in Tucker's Lemma. The way to picture $R_{p,m}$ is as follows: the point 0 lies at the center and there are p segments of length m leaving from 0 in p different directions. In that sense, we also call $R_{p,m}$ a p-star. The boundary of the p-star is the set of points $*^1 m, \ldots, *^p m$.

The $\mathbb{Z}_p$-action θ is defined on $R_{p,m}$ in the natural way, i.e., $\theta(*^i j) = *^{i+1} j$ (recall that $i \in \mathbb{Z}_p$). In particular, $\theta(0) = 0$. For any $d \geq 1$, θ can be extended to $R^d_{p,m} := (R_{p,m})^d$ by simply applying θ separately to each coordinate. Note that θ is a free action when restricted to the boundary of $R^d_{p,m}$ (i.e., the points that have at least one coordinate of the form $*^{\cdot} m$). See Figure 6.

There is a very natural metric on $R_{p,m}$. $\mathrm{dist}(*^{i_1} j_1, *^{i_2} j_2)$ is defined to be $|j_1 - j_2|$ if $i_1 = i_2$, and $j_1 + j_2$ otherwise. We let $\mathrm{dist}_\infty(\cdot, \cdot)$ denote the generalization to $R^d_{p,m}$ (where we take the maximum). Finally, we triangulate the domain $R^d_{p,m}$ by using Kuhn's triangulation on every subcube (see Remark 5 below for details). We can now state a Tucker's lemma for this domain.

Theorem 5.1 ($\mathbb{Z}_p$-star Tucker's Lemma)**.** *Let p be prime and $m, t \geq 1$, $d = t(p-1)$, and T be Kuhn's triangulation of $R^d_{p,m}$. Let $\lambda : R^d_{p,m} \to R_{p,t} \setminus \{0\}$[4] be any labeling that satisfies $\lambda(\theta x) = \theta \lambda(x)$ for all*

[4]Notice that the set $R_{p,t} \setminus \{0\}$ is isomorphic to the set $[p] \times [t]$.

$x \in \partial R^d_{p,m}$. Then there exists a $(p-1)$-simplex $x_1, \dots, x_p$ of T and $j \in [t]$ such that $\lambda(x_i) = *^i j$ for all $i \in [p]$.

In particular, by the properties of Kuhn's triangulation, it holds that for every solution $x_1, \dots, x_p$, we have $\mathrm{dist}_\infty(x_i, x_k) \leq 1$ for all $i, k \in [p]$.

Note that $\mathbb{Z}_2$-star Tucker's Lemma corresponds to the standard version of Tucker's Lemma. Since we are interested in the computational aspect, we also define the naturally corresponding TFNP problem.

<u>$\mathbb{Z}_p$-STAR-TUCKER:</u>

INPUT: $m, t \geq 1$, $d = t(p-1)$ and a Boolean circuit computing a labeling $\lambda : R^d_{p,m} \to R_{p,t} \setminus \{0\}$ that satisfies $\lambda(\theta x) = \theta\lambda(x)$ for all $x \in \partial R^d_{p,m}$

OUTPUT: A $(p-1)$-simplex $x_1, \dots, x_p \in R^d_{p,m}$ and $j \in [t]$ such that $\lambda(x_i) = *^i j$ for all $i \in [p]$

Note that the property "$\lambda(\theta x) = \theta\lambda(x)$ for all $x \in \partial R^d_{p,m}$" can be enforced syntactically. Thus, $\mathbb{Z}_p$-STAR-TUCKER is not a promise problem.

The main result of the section is the following theorem.

Theorem 5.2. *For all primes p, $\mathbb{Z}_p$-STAR-TUCKER is* PPA-*p-complete.*

The proof of the theorem will follow from Theorem 5.3 and Proposition 5.4 below. The hardness is obtained by a reduction from p-BSS-TUCKER , which is PPA-p-hard for all primes p, as shown in Theorem 4.10.

Remark 5. The domain $R^d_{p,m}$ can be triangulated in a standard way as follows. We start by subdividing the domain into hypercubes $\{*^{i_1}(a_1 - 1), *^{i_1}a_1\} \times \cdots \times \{*^{i_d}(a_d - 1), *^{i_d}a_d\}$ for $a_1, \dots, a_d \in [m]$ and $i_1, \dots, i_d \in \mathbb{Z}_p$. Then, we can use Kuhn's triangulation on each hypercube.

Similarly to Remark 4, the triangulation T of $R^d_{p,m}$ has the following nice properties:

1. The restriction of T on any sub-orthant of $R^d_{p,m}$ (i.e., a subspace of the form $A_1 \times A_2 \times \cdots \times A_d$, where $A_\ell = \{*^{i_\ell} j : 0 \leq j \leq m\}$ or $A_\ell = \{0\}$) yields a triangulation of that sub-orthant.

2. On the boundary of $R^d_{p,m}$, the triangulation T is symmetric with respect to θ : for any simplex σ of T that lies on the boundary of $R^d_{p,m}$, the simplices $\theta\sigma, \theta^2\sigma, \dots, \theta^{p-1}\sigma$ are also simplices of T (that also lie on the boundary).

3. T is computationally efficient, in the sense that we can perform pivoting and indexing operations in polynomial time.

5.1 $\mathbb{Z}_p$-STAR-TUCKER is in PPA-p

First, we prove the membership of $\mathbb{Z}_p$-STAR-TUCKER in PPA-p. We have the following theorem.

Theorem 5.3. *For all primes p, $\mathbb{Z}_p$-STAR-TUCKER lies in* PPA-*p.*

The result is proved by reducing the problem to IMBALANCE-MOD-p. In particular, this also provides a combinatorial proof of $\mathbb{Z}_p$-star Tucker's Lemma (Theorem 5.1). This proof can be seen as a generalization of the combinatorial proof of Tucker's Lemma given by Freund and Todd [1981]. We provide an overview of the proof below; the full proof can be found in Appendix C.

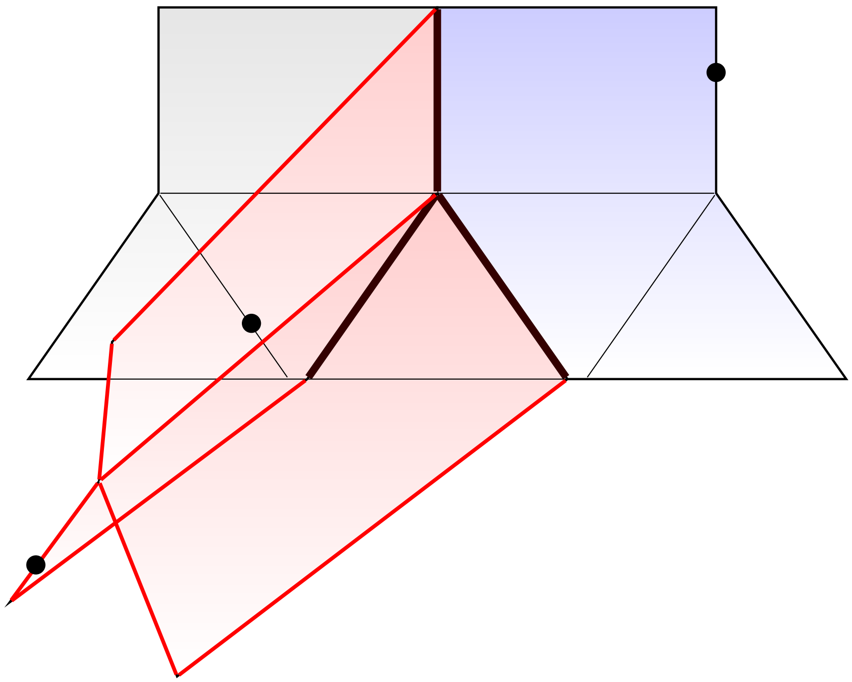

Figure 6: A view of the domain $R^d_{p,m}$ for $p = 3$ and $d = 2$. Note that this corresponds to $R_{3,m} \times R_{3,m}$. The three black points are in correspondence under θ. The three thick lines at the center of the picture correspond to the place where the three pieces are "glued" together.

5.1.1 Proof Overview of Theorem 5.3

As noted earlier, $\mathbb{Z}_2$-STAR-TUCKER corresponds to the standard version of Tucker's Lemma and the domain is equivalent to $\{-m, -(m-1) \ldots, 0, \ldots, m\}^d$. The computational problem is known to lie in PPA (recall that PPA = PPA-2) by using an argument given by Freund and Todd [1981]. More precisely, Freund and Todd gave a constructive proof of Tucker's Lemma and as noted by [Papadimitriou, 1994, Aisenberg et al., 2020], this yields a reduction to PPA. The constructive proof relies on a path-following argument on a graph where the nodes are simplices of the triangulation. We start by giving some details about their argument, since our proof is a generalization of their construction.

The nodes of the graph G consist of all simplices of the triangulation that satisfy some properties that depend on the labels of the simplex (and the coordinate subspace orthant in which the simplex lies). Following the presentation of the proof given by Matoušek [2008], we call these "happy simplices". Undirected edges are added between happy simplices based on some simple rules (e.g., if they share a facet and that facet has some desired labels, etc). Given the definition of the edges, it is easy to show that the happy simplex 0^d has degree 1 and any other node of degree 1 is either a solution, or is a happy simplex lying on the boundary of the domain. In the latter case, because of the boundary conditions, this means that there must be another such simplex that lies on the antipodally opposite side of the domain and also has degree 1. Thus, merging these two nodes into a single one yields a vertex of degree 2, eliminating these "fake" solutions.

Our first contribution is to note that the edges of the graph can be directed in a *consistent* way in Freund and Todd's construction. Namely, any non-merged degree-2 vertex has one incoming and one outgoing edge, and any merged vertex has either two incoming edges, or two outgoing edges. This yields a reduction to Imbalance-mod-2. Note that if all degree 2 vertices were always perfectly balanced, then we would obtain a reduction to End-of-Line, which is impossible, unless PPAD = PPA.

When we move to the case $p > 2$, the notions used to define the graph can be generalized in a natural way, despite the unusual domain $R^d_{p,m}$. While the ability to direct edges was not actually needed for $p = 2$, it now becomes absolutely necessary. Indeed, for any degree-1 happy simplex on the boundary, there are now $p - 1$ other such simplices (by using θ). Merging these into a single node yields degree p. We show that directing the edges yields a merged node that is balanced modulo p (namely, all p edges are incoming, or they are all outgoing).

However, another difficulty arises for $p > 2$. Recall that a path can visit simplices of various dimensions. The vertices where the dimension changes are special vertices, that we call super-happy simplices. These super-happy simplices have one edge with a same or lower-dimensional happy simplex, and k edges with k different higher-dimensional happy-simplices, where $k \in [p-1]$. Directing the edges as before, yields that the k edges are directed the same way, and in the opposite direction to the single edge. By changing the way the direction of edges is defined, it is possible to salvage the situation for $p = 3$. However, this fails for any $p \geq 5$.

The solution is to carefully assign *weights* to all edges. The weight of an edge only depends on the nature of the coordinate subspace orthants in which it lies, in particular the dimension. With these weights, we show that any vertex that is not a solution is now balanced modulo p (except the trivial solution 0^d). Namely, the non-solution vertices of the graph are:

- the trivial solution 0^d: all its edges are outgoing and it has degree $(-1)^t \mod p$
- the merged simplices on the boundary: p edges, all incoming/outgoing, all the same weight
- happy, but not super-happy simplices: once incoming edge, one outgoing, both same weight
- super-happy simplices: one incoming edge with weight w and $k \in [p-1]$ outgoing edges, each with weight w/k (or opposite direction for all edges)

Thus, apart from the trivial solution and any actual solutions, all vertices are balanced modulo p.

Remark 6 (Path-following arguments). Even though this proof is a natural generalization of the argument by Freund and Todd [1981], it is not a *path-following argument* for $p \geq 3$. Indeed, in the case where $p \geq 3$, it is not clear how we could explore this graph by following a path that is guaranteed to end at a solution. In fact, we provide strong evidence that it is not possible to prove $\mathbb{Z}_p$-star-Tucker by a path-following argument. Since $\mathbb{Z}_p$-star-Tucker is PPA-p-hard (see next Section), and since a path-following proof of $\mathbb{Z}_p$-star-Tucker would presumably show that the problem lies in PPA, this would imply that PPA-$p \subseteq$ PPA, for prime $p \geq 3$. However, this is not expected to hold [Johnson, 2011, Göös et al., 2020, Hollender, 2019].

Similarly, if one can show that p-Necklace-Splitting is PPA-p-hard for some prime $p \geq 3$, then this would provide strong evidence that the Necklace Splitting theorem with p thieves cannot be proved by a path-following argument.

5.2 $\mathbb{Z}_p$-star-Tucker is PPA-p-hard

In this section we prove that $\mathbb{Z}_p$-star-Tucker is PPA-p-hard, by reducing from p-BSS-Tucker, which is PPA-p-hard by Theorem 4.10.

Proposition 5.4. *For any prime p, $\mathbb{Z}_p$-*STAR*-*TUCKER *is* PPA-*p-hard, even for fixed dimension $t \geq 1$.*

Proof. We show this by reducing from p-BSS-TUCKER with $n = t$. Let λ be an instance of p-BSS-TUCKER for some prime p and some $n \geq 1$, where we use Kuhn's triangulation.

In this case the domain of p-BSS-TUCKER corresponds to

$$C_\infty = \{(c^1, \ldots, c^p) \in U_m^{np} | \forall j \in [n], \exists i \in [p] : c_j^i = 0\}.$$

On the other hand, the domain of $\mathbb{Z}_p$-STAR-TUCKER can be described as

$$A = \{(a^1, \ldots, a^p) \in U_m^{np(p-1)} | \forall j, k \in [n] \times [p-1], \exists i^* \in [p] : a_{j,k}^i = 0 \text{ for all } i \in [p] \setminus \{i^*\}\}.$$

Note that $\theta(c^1, \ldots, c^p) = (c^p, c^1, \ldots, c^{p-1})$ and $\theta(a^1, \ldots, a^p) = (a^p, a^1, \ldots, a^{p-1})$.

Define $\Psi : A \to C_\infty$ as $\Psi(a^1, \ldots, a^p) = (\psi(a^1), \ldots, \psi(a^p))$, where $\psi_j(a^i) = \max\{a_{j,k}^i : k \in [p-1]\}$ for all $i \in [p]$, $j \in [n]$. Note that Ψ is well-defined, namely if $a \in A$, then $\Psi(a) \in C_\infty$. Indeed, for any $j \in [n]$, it holds that $|\{i \in [p] | \psi_j(a^i) > 0\}| = |\{i \in [p] | \exists k \in [p-1] : a_{j,k}^i > 0\}| \leq |\{(i,k) \in [p] \times [p-1] | a_{j,k}^i > 0\}| \leq p - 1$, since for every $k \in [p-1]$ there exists at most one $i \in [p]$ such that $a_{j,k}^i > 0$ (for any fixed j). Furthermore, it is easy to see that $\Psi(\theta a) = \theta \Psi(a)$ by construction.

Now define the labeling $\lambda' : A \to \mathbb{Z}_p \times [n]$ by $\lambda'(a) := \lambda(\Psi(a))$. Since $\Psi(A) \subseteq C_\infty$, λ' is well-defined. Furthermore, for any $a \in \partial A$, it holds that $\Psi(a) \in \partial C_\infty$. Thus, we get that for any $a \in \partial A$, $\lambda'(\theta a) = \lambda(\Psi(\theta a)) = \lambda(\theta \Psi(a)) = \theta \lambda(\Psi(a)) = \theta \lambda'(a)$, i.e., λ' satisfies the boundary conditions.

If $\sigma = \{z_1, \ldots, z_p\}$ is a $(p-1)$-simplex of A such that $\lambda'(\sigma) = \mathbb{Z}_p \times \{j\}$ for some $j \in [n]$, then the set of vertices $\Psi(\sigma) = \{\Psi(z_1), \ldots, \Psi(z_p)\}$ satisfies $\lambda(\Psi(\sigma)) = \mathbb{Z}_p \times \{j\}$. Thus, the proof is completed by the following claim:

Claim 1. If $\sigma = \{z_1, \ldots, z_p\}$ is a $(p-1)$-simplex in Kuhn's triangulation of A, then $\Psi(\sigma) = \{\Psi(z_1), \ldots, \Psi(z_p)\}$ is a simplex in Kuhn's triangulation of C_∞.

Proof. Since we use Kuhn's triangulation, without loss of generality, we can assume that $z_1, \ldots, z_p$ are ordered such that $z_1 \leq z_2 \leq \cdots \leq z_p$ (component-wise) and $\|z_1 - z_p\|_\infty \leq 1/m$. By construction of Ψ it holds that $\Psi(a) \leq \Psi(a')$ whenever $a \leq a'$. Thus, $\Psi(z_1) \leq \cdots \leq \Psi(z_p)$. In order to show that $\Psi(\sigma)$ is a simplex in Kuhn's triangulation of C_∞, it remains to prove that $\|\Psi(z_1) - \Psi(z_p)\|_\infty \leq 1/m$. To see this, note that if $\|a - a'\|_\infty \leq 1/m$, then for all $i \in [p]$ and $j \in [n]$, we get that $|\psi_j(a^i) - \psi_j(a'^i)| = |\max\{a_{j,k}^i : k \in [p-1]\} - \max\{a'^i_{j,k} : k \in [p-1]\}| \leq 1/m$. ☐

This concludes the proof of Proposition 5.4. ☐

6 Necklace Splitting with p Thieves lies in PPA-p

In this section, we prove our main result regarding the Necklace Splitting Theorem; we prove that the associated computational problem p-NECKLACE-SPLITTING lies in PPA-p for any prime p.

Theorem 6.1. *For every prime p, p-*NECKLACE-SPLITTING *is in* PPA-*p.*

As a corollary, we also obtain that:

- p^r-NECKLACE-SPLITTING is in PPA-p^r for any prime p and $r \geq 1$

- k-Necklace-Splitting lies in PPA-k *under Turing reductions* for any $k \geq 2$.

As we explained in the introduction, our reduction will go via the continuous version of the problem, the ε-Consensus-$1/p$-Division problem, which is the computational analogue of the Consensus-$1/p$-Division problem of Simmons and Su [2003]. We will show the inclusion of ε-Consensus-$1/p$-Division in PPA-p via a reduction to $\mathbb{Z}_p$-star-Tucker, which also implies the PPA-p membership for p-Necklace-Splitting. We prove the following main statement:

Theorem 6.2. *For any prime p,* Consensus-$1/p$-Division *reduces to* $\mathbb{Z}_p$-star-Tucker.

The proof is presented in Section 6.1, where we prove an even stronger version of this theorem. Indeed, we show that this result holds for any probability measures (not only step functions), as long as they are efficiently computable and sufficiently continuous (in some precise sense).

We start with the complete definitions of the computational problems corresponding to k-thief Necklace Splitting and Consensus-$1/k$-Division.

<u>k-Necklace-Splitting</u> [Papadimitriou, 1994, Filos-Ratsikas and Goldberg, 2019]

Input: An open necklace with n beads, each of which has one of t colors.
There are exactly $a_i k$ beads of color $i = 1, \ldots, t$, where $a_i \in \mathbb{N}$.

Output: A partitioning of the necklace into k (not necessarily connected) pieces such that each piece contains exactly a_i beads of color i, using at most $(k-1)t$ cuts.

As proved by Alon [1987], this problem always has a solution. Furthermore, for any proposed splitting of the necklace it is easy to check if it is a solution. Thus, the problem k-Necklace-Splitting lies in TFNP.

Alon [1987] proved the necklace-splitting theorem by showing existence for a more general continuous version and then "rounding" a solution of the continuous problem to obtain a splitting of the necklace. When investigating the complexity of k-Necklace-Splitting, it is also convenient to consider the more general continuous version. Even though the continuous theorem is termed as a "generalized Hobby-Rice theorem" by Alon [1987], we instead use the term *Consensus-$1/k$-Division* proposed by Simmons and Su [2003].

<u>ε-Consensus-$1/k$-Division</u> [Filos-Ratsikas et al., 2018, Filos-Ratsikas and Goldberg, 2018]

Input: $\varepsilon > 0$ and continuous probability measures $\mu_1, \ldots, \mu_t$ on $[0, 1]$.
The probability measures are given by their density functions on $[0, 1]$, which are step functions (explicitly given in the input).

Output: A partitioning of the unit interval into k (not necessarily connected) pieces $A_1, \ldots, A_k$ using at most $(k-1)t$ cuts, such that $|\mu_j(A_i) - \mu_j(A_\ell)| \leq \varepsilon$ for all i, j, ℓ.

The fact that this problem also lies in TFNP immediately follows from showing that it lies in PPA-k under Turing reductions (Theorem 6.4). Note that we can equivalently ask for $\mu_j(A_i) = 1/k \pm \varepsilon$ for all i, j. Indeed, the computational problems are equivalent.

Furthermore, using the same technique as Etessami and Yannakakis [2010, Theorem 5.2], one can show that if ε is sufficiently small (with respect to the representation size of the step functions), then one can efficiently compute an *exact* solution from an ε-approximate solution. It follows that ε-Consensus-1/k-Division is equivalent to exact Consensus-1/k-Division. In particular, the problem always has an exact solution that is rational. Thus, we will sometimes refer to this problem just as Consensus-1/k-Division.

Alon's rounding procedure yields a reduction from k-Necklace-Splitting to exact Consensus-1/k-Division. Filos-Ratsikas and Goldberg [2018] extended this result by showing that k-Necklace-Splitting reduces to ε-Consensus-1/k-Division, even when ε is not small enough to ensure that we can get an exact solution.

Proposition 6.3 (Alon [1987], Filos-Ratsikas and Goldberg [2018])**.** *For any $k \geq 2$, k-*Necklace-Splitting *reduces to ε-*Consensus-1/k-Division*.*

Before we proceed with the proof of Theorem 6.2, we present the consequences of Theorems 5.3 and 6.2, in terms of computational complexity and mathematical existence.

Consequences: Computational Complexity

Theorem 6.4. *For any $k \geq 2$, k-*Necklace-Splitting *and* Consensus-1/k-Division *lie in the Turing closure of* PPA-k*. In particular, if $k = p^r$ where p is prime and $r \geq 1$, then the problems lie in* PPA-p*.*

The Turing closure of PPA-k is the class of all TFNP problems that Turing-reduce to a PPA-k-complete problem (e.g., Imbalance-mod-k). Note that when k is not a prime power, PPA-k is not believed to be closed under Turing reductions [Göös et al., 2020, Hollender, 2019].

Proof. Theorem 5.3 and Theorem 6.2 immediately imply that for any prime p, Consensus-1/p-Division lies in PPA-p. As noted by Alon [1987, Proposition 3.2], for any $k, \ell \geq 2$, a Consensus-1/$(k\ell)$-Division can be obtained by first finding a Consensus-1/k-Division – which divides the interval into k (not necessarily connected) pieces – and then finding a Consensus-1/ℓ-Division of each of the k pieces. Note that we obtain (at most) the desired number of cuts. Thus, we can solve an instance of Consensus-1/$(k\ell)$-Division by first solving an instance of Consensus-1/k-Division, and then k instances of Consensus-1/ℓ-Division.

In particular, for any prime p and any $r \geq 1$, Consensus-1/p^r-Division can be solved by solving $1 + p + p^2 + \cdots + p^{r-1}$ instances of Consensus-1/p-Division. Thus, we obtain a Turing reduction from Consensus-1/p^r-Division to Consensus-1/p-Division. Since Consensus-1/p-Division lies in PPA-p and PPA-p is closed under Turing reductions [Göös et al., 2020, Hollender, 2019], it follows that Consensus-1/p^r-Division lies in PPA-p.

Now consider any $k = \prod_{i=1}^m p_i^{r_i}$, where $m \geq 1$, $r_i \geq 1$ and the p_i are distinct primes. Then, Consensus-1/k-Division can be solved by a query to Consensus-1/$p_1^{r_1}$-Division, then $p_1^{r_1}$ queries to Consensus-1/$p_2^{r_2}$-Division (which can be turned into a single query to PPA-p_2), then $p_1^{r_1} p_2^{r_2}$ queries to Consensus-1/$p_3^{r_3}$-Division (which can also be turned into a single query to PPA-p_3), etc. Thus, Consensus-1/k-Division can be solved by a query to PPA-p_1, then a query to PPA-p_2, then one to PPA-p_3, $\ldots$, and finally a query to PPA-p_m. Since PPA-$p_i \subseteq$ PPA-k for $i = 1, \ldots, m$ (Proposition 2.2), it follows that there is a Turing reduction from Consensus-1/k-Division to a PPA-k-complete problem (e.g., Imbalance-mod-k).

Since k-Necklace-Splitting reduces to Consensus-1/k-Division (Proposition 6.3), the results also hold for k-Necklace-Splitting. □

Consequences: Mathematical Existence

Theorems 5.3 and 6.2 yield a reduction from Consensus-$1/p$-Division to Imbalance-mod-p. Since every instance of Imbalance-mod-p has a solution (and the proof of this is trivial), we obtain a proof that Consensus-$1/p$-Division always has a solution. Thus, this proves that if the probability measures are step functions (described by rational numbers), there always exists a consensus-$1/p$-division. While we have given the proof in terms of a reduction (since this is required for our complexity results), it can also be written as a mathematical proof of existence (without any computational considerations).

Once existence of a consensus-$1/p$-division for step functions has been proved, a constructive argument by Alon [1987, Section 2] also gives existence for p-necklace-splitting. Putting everything together, the proof of p-necklace-splitting thus obtained is a fully combinatorial proof that does not use any advanced machinery and is easier to follow than existing proofs. Indeed, as we mentioned in the introduction, the original proof by Alon [1987] used the BSS theorem of Bárány et al. [1981] as a black box. The only other fully combinatorial proof by Meunier [2014], while quite elegant, is significantly more involved.

Going back to consensus-$1/p$-division, the proof we obtain (which uses $\mathbb{Z}_p$-star Tucker's lemma) actually works for any probability measures, not only step functions. Moreover, similarly to Simmons and Su [2003], our proof does not make use of the fact that the measures are additive and non-negative. Thus, we obtain a stronger version of the consensus-$1/p$-division theorem given by Alon [1987, Theorem 1.2] (which he calls a generalization of the Hobby-Rice theorem). Let $\mathcal{B}([0,1])$ denote the Borel σ-algebra on the unit interval and let λ denote the Lebesgue measure on the unit interval. Finally, let $\triangle$ denote the symmetric difference, i.e., $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Theorem 6.5. *Let p be any prime and $t \geq 1$. Let $v_1, \dots, v_t : \mathcal{B}([0,1]) \to \mathbb{R}$ be such that for all $1 \leq j \leq t$ v_j satisfies the following continuity condition: for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|v_j(A) - v_j(B)| \leq \varepsilon \quad \text{for all } A, B \in \mathcal{B}([0,1]) \text{ that satisfy } \lambda(A \triangle B) \leq \delta.$$

Then, there exists a consensus-$1/p$-division. Namely, it is possible to partition the unit interval into p (not necessarily connected) pieces $A_1, \dots, A_p$ using at most $(p-1)t$ cuts, such that $v_j(A_i) = v_j(A_\ell)$ for all $1 \leq i, \ell \leq p$, $1 \leq j \leq t$.

Before we move on to the proof, let us briefly explain why we only obtain the result for prime p. In the usual setting where the valuations are probability measures, it is enough to prove the statement for primes. Indeed, using the standard argument by Alon [1987, Proposition 3.2], if a consensus-$1/k$-division and a consensus-$1/\ell$-division always exist, then a consensus-$1/(k\ell)$-division exists. But Alon's argument makes use of the additivity property of the measures. Indeed, consider the non-additive setting and say that we are trying to show that a consensus-1/4-exists. We know that a consensus-1/2-division exists and this yields a partition of $[0,1]$ into A_1 and A_2. We have that $v_j(A_1) = v_j(A_2)$ for all j. Following Alon's argument, find a consensus-1/2-division of A_1 and of A_2. This yields $A_{11} \cup A_{12} = A_1$ and $A_{21} \cup A_{22} = A_2$ such that $v_j(A_{11}) = v_j(A_{12})$ and $v_j(A_{21}) = v_j(A_{22})$ for all j. However, we might not have $v_j(A_{11}) = v_j(A_{22})$, since we no longer have $v_j(A_{11}) + v_j(A_{12}) = v_j(A_{21}) + v_j(A_{22})$.

If we assume that the valuations are additive (even just finite additivity), but still allow them to take negative values, then Alon's argument works as before and thus the result again holds for any $k \geq 2$.

Proof. Using $\mathbb{Z}_p$-star Tucker's lemma (Theorem 5.1) it follows that for any $\varepsilon > 0$, there exists an ε-approximate consensus-$1/p$-division, i.e., we can partition the interval into p pieces $A_1, \dots, A_p$ (using at most $(p-1)t$ cuts) such that $|v_j(A_i) - v_j(A_\ell)| \le \varepsilon$ for all $1 \le i, \ell \le p$, $1 \le j \le t$. Indeed, it suffices to follow the same steps as in the proof of Theorem 6.6.

Let R_p denote the continuous version of $R_{p,m}$. R_p consists of p copies of the segment $[0,1]$ (namely $*^1[0,1]$, $\dots$, $*^p[0,1]$) that share the same origin (i.e., $*^1 0 = \cdots = *^p 0$). Using the interpretation given in the proof of Theorem 6.6, every point in R_p^d corresponds to a way to partition $[0,1]$ into p (not necessarily connected) pieces using at most $(p-1)t$ cuts. Since $(R_p^d, \mathrm{dist}_\infty)$ is a compact metric space, every sequence must have a subsequence that converges. Thus, a sequence $(x_n)_n$, where $x_n \in R_p^d$ is a $1/2^n$-approximate consensus-$1/p$-division, must have a subsequence that converges to some $x \in R_p^d$. For any i, j the function $f : R_p^d \to \mathbb{R}$, $y \mapsto v_j(A_i(y))$ is continuous. It follows that x must correspond to an exact consensus-$1/p$-division. $\square$

6.1 Reduction from Consensus-$1/p$-Division to $\mathbb{Z}_p$-star-Tucker

In order to make our PPA-p-membership result as strong as possible, we define a computational problem that is much more general than ε-Consensus-$1/p$-Division. Namely, we allow any computationally reasonable probability measures that are also sufficiently continuous.

The probability measures are given by their cumulative functions. Let $\mathcal{F}$ be a class of cumulative distribution functions on $[0,1]$. Thus, for any $f \in \mathcal{F}$ and any $a \in [0,1]$, $f(a)$ is the probability of the interval $[0,a]$ according to f. For any $f \in \mathcal{F}$, let $\mathrm{size}(f)$ denote the size of the representation of f. E.g., if f is represented as a circuit, then $\mathrm{size}(f)$ is the size of the circuit. For any rational number x, $\mathrm{size}(x)$ denotes the representation length of x, i.e., the length of the binary representation of the denominator and numerator of x. We will require two properties from $\mathcal{F}$:

- $\mathcal{F}$ is polynomially computable: there exists a polynomial q_1 such that for all $f \in \mathcal{F}$ and all rational $x \in [0,1]$, $f(x)$ can be computed in time $q_1(\mathrm{size}(f) + \mathrm{size}(x))$.
- $\mathcal{F}$ is polynomially continuous: there exists a polynomial q_2 such that for all $f \in \mathcal{F}$ and all rational $\widehat{\varepsilon} > 0$, there exists rational $\widehat{\delta} > 0$ with $\mathrm{size}(\widehat{\delta}) \le q_2(\mathrm{size}(f) + \mathrm{size}(\widehat{\varepsilon}))$ such that $|x - y| \le \widehat{\delta} \implies |f(x) - f(y)| \le \widehat{\varepsilon}$ for all $x, y \in [0,1]$.

These properties are quite natural and they were used by Etessami and Yannakakis [2010] in the context of fixed point problems. In particular, they hold when $\mathcal{F}$ is the class of all cumulative distribution functions given by step function densities (represented explicitly). But they also hold for much more general families.

Definition 10. Let $k \ge 2$ and let $\mathcal{F}$ be a polynomially computable and polynomially continuous class of cumulative distribution functions on $[0,1]$. The problem ε-Consensus-$1/k$-Division[$\mathcal{F}$] is defined exactly as ε-Consensus-$1/k$-Division, except that the probability measures are given by cumulative distribution functions in $\mathcal{F}$.

Notice that ε-Consensus-$1/k$-Division corresponds to the special case where $\mathcal{F}$ is the class of all cumulative distribution functions given by step function densities (represented explicitly). Thus, the following is a stronger version of Theorem 6.2.

Theorem 6.6. *Let p be prime and $\mathcal{F}$ be a polynomially computable and polynomially continuous class of cumulative distribution functions on $[0,1]$. Then ε-Consensus-$1/p$-Division[$\mathcal{F}$] reduces to $\mathbb{Z}_p$-star-Tucker.*

Proof. Let $\varepsilon > 0$ and $\mu_1, \ldots, \mu_t$ be probability measures on $[0,1]$ given by functions in $\mathcal{F}$. We consider the domain $D = R^d_{p,m}$, where $d = t(p-1)$ and $m \geq 1$ will be set later. A point in D represents a way to partition $[0,1]$ into p (not necessarily connected) pieces using at most $t(p-1)$ cuts. This is a slight modification of the domain that was used by Meunier [2014] to encode a splitting of a necklace. Intuitively it can be explained as follows. Let $x = (*^{i_1} j_1, \ldots, *^{i_d} j_d) \in D$. We interpret each element $i \in \mathbb{Z}_p$ as a different color. Then:

1. Paint the whole interval $[0,1]$ with the color 1.

2. For $\ell = 1, 2, \ldots, d$: paint $[0, j_\ell/m]$ with the color i_ℓ

Note that applying a fresh coat of paint on a previously painted part of the interval covers up the old paint. The way the interval $[0,1]$ is colored at the end of this procedure gives us the partition encoded by $x \in D$. An important advantage of this encoding is that it is sufficiently continuous in a certain sense. Indeed, small changes in the coordinates of x have a small effect on the corresponding partition. Other simpler encoding schemes do not have this property.

Formally, this encoding can be described as follows. Add a "fake" 0th coordinate $*^{i_0} j_0 = *^1 m$. Place cuts at all positions $j_0/m, j_1/m, \ldots, j_d/m$. This subdivides the interval $[0,1]$ into at most $d + 1 = t(p-1) + 1$ subintervals. Then, allocate the subinterval $[a,b]$ to $i_{\widehat{\ell}} \in \mathbb{Z}_p$, where $\widehat{\ell} = \max\{0 \leq \ell \leq d : j_\ell/m \geq b\}$.

This encoding also behaves nicely with respect to θ. For any $x \in \partial D$, θx encodes the same partition as x, except that i has been replaced by $i+1$, for all $i \in \mathbb{Z}_p$. This is easy to see since for any $x \in \partial D$, there exists $\ell \geq 1$ such that $j_\ell = m$ and thus the "fake" coordinate $*^{i_0} j_0$ does not play any role.

We are now ready to define the labeling $\lambda : D \to R_{p,t} \setminus \{0\}$. This labeling is a natural generalization of the one used by Simmons and Su [2003]. Given $x \in D$, construct the partition it encodes, namely $A_1(x), \ldots, A_p(x)$. Then, for all $i \in \mathbb{Z}_p$ and $j \in [t]$, let $\mu_{j,i}(x) = \mu_j(A_i(x))$, i.e., the total measure of type j that is allocated to i. Finally, set $\lambda(x) = *^i j$, where i, j are determined as follows:

1. Pick $j \in [t]$ that maximizes $\max_{i_1,i_2} |\mu_{j,i_1}(x) - \mu_{j,i_2}(x)|$. Break ties by picking the minimum such j.

2. Then, pick $i \in \mathbb{Z}_p$ that maximizes $\mu_{j,i}(x)$. If there are multiple i's that maximize this, break ties by picking the one such that $\min A_i(x)$ is minimal (i.e., such that $A_i(x)$ contains the point closest to the left end of the unit interval).

By using the observation above about the behavior of θ on ∂D, it is easy to see that $\lambda(\theta x) = \theta\lambda(x)$ for all $x \in \partial D$. Thus, λ is a valid instance of $\mathbb{Z}_p$-STAR-TUCKER and we obtain a solution $x_1, \ldots, x_p \in D$ and $\widehat{\jmath} \in [t]$, such that $\mathrm{dist}_\infty(x_i, x_k) \leq 1$ and $\lambda(x_i) = *^i \widehat{\jmath}$ for all $i, k \in [p]$. It remains to show that by picking m large enough, we obtain a solution to ε-CONSENSUS-$1/p$-DIVISION[$\mathcal{F}$].

Let $\varepsilon' = \frac{\varepsilon}{2t(p-1)}$. Since $\mathcal{F}$ is polynomially continuous, we can pick m large enough so that the value $\mu_j([0,a])$ changes by at most ε', if a moves by $1/m$. Note that m has representation length polynomial in the size of the instance. If a single coordinate of x changes by 1, $\mu_{j,i}(x)$ changes by at most ε' for all i, j. Since there are $d = t(p-1)$ coordinates, it follows that $|\mu_{j,i}(x_k) - \mu_{j,i}(x_\ell)| \leq \varepsilon' t(p-1) = \varepsilon/2$ for all i, j and for all $k, \ell \in [p]$.

Let $x := x_1$. By construction of the labeling, we obtain that for all j, i, ℓ

$$|\mu_{j,i}(x) - \mu_{j,\ell}(x)| \leq \max_{i_1,i_2} |\mu_{\widehat{\jmath},i_1}(x) - \mu_{\widehat{\jmath},i_2}(x)| \leq \varepsilon$$

The first inequality holds because $\lambda(x) = *^1\widehat{\jmath}$. The second inequality holds because if we instead had $\mu_{\widehat{\jmath},i_1}(x) > \mu_{\widehat{\jmath},i_2}(x) + \varepsilon$ for some i_1, i_2, then it would follow that $\mu_{\widehat{\jmath},i_1}(x_{i_2}) > \mu_{\widehat{\jmath},i_2}(x_{i_2})$, contradicting $\lambda(x_{i_2}) = *^{i_2}\widehat{\jmath}$. Thus, x corresponds to an ε-approximate solution. □

7 Conclusion and Future Work

Our topological characterization of PPA-p can possibly enable us to obtain similar membership or hardness results for other interesting problems. For example, are the problems whose totality is established via the BSS Theorem, like the Chromatic Number of Kneser Hypergraphs[5] studied in [Alon et al., 1986] in PPA-p? Are they PPA-p-complete? We believe that due to its simplicity, our p-POLYGON-TUCKER problem can be a very useful tool for obtaining hardness results for these problems. What about other problems that generalize problems that are known to be in PPA or are even PPA-complete? For example, Filos-Ratsikas and Goldberg [2019] showed that the discrete Ham-Sandwich problem is also PPA-complete. Is there a generalization of the problem that could be complete for PPA-p? A computational version of the Center Transversal Theorem [Dol'nikov, 1992, Zivaljević and Vrećica, 1990] might be a good candidate. Another interesting open problem is to investigate the connection of the general statement of Dold's Theorem [Dold, 1983] from algebraic topology with the subclasses of TFNP. Finally, although our paper takes a definitive step in the direction of resolving the complexity of p-thief Necklace Splitting and Consensus-$1/p$-Division, proving a PPA-p-hardness result remains a challenging open problem. In very recent work [Filos-Ratsikas et al., 2020], we have made a first step in that direction, by providing a significantly simpler proof (and strengthening) of the PPA-2-hardness result of Filos-Ratsikas and Goldberg [2019], as well as the first hardness result for Consensus-1/3-Division, showing that it is hard for the class PPAD. Showing the PPAD-hardness of the Consensus-$1/p$-Division problem for $p > 3$ is also a very interesting first step that might be easier than showing the PPA-p-hardness.

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A Topological Definitions

In this section, we include all the necessary notation and topological definitions that are used throughout the paper. For a more detailed exposition on simplicial complexes, we refer the interested reader to [Matoušek, 2008].

Notation: Let $B^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ denote the n-dimensional unit ball and $S^{n-1} = \partial B^n$ be the corresponding unit sphere.

Definition 11 (Homeomorphism)**.** A *homeomorphism* of topological spaces $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ is a bijection $\phi : X_1 \to X_2$ such that for every $U \subseteq X_1$, $\phi(U) \in \mathcal{O}_2$ if and only if $U \in \mathcal{O}_1$. In other words, a bijection $\phi : X_1 \to X_2$ is a homeomorphism if and only if both ϕ and ϕ^{-1} are continuous. If there is a homeomorphism $\phi : X_1 \to X_2$, we write $X \cong Y$.

We say that a function f has *order* p if $f^p = f$, where the notation f^i denotes to the composition of f by itself i times.

Definition 12 (Free Action)**.** Let $f : X \to Y$ be a function of order p and let P be a set. We say that f acts freely on P if for all $x \in X$ and all $i \in \{1, \ldots, p-1\}$, $f^i(x) \neq x$.

Definition 13 (Affine Independence)**.** We call the points $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *affine dependent* if there exist numbers $a_1, a_2, \ldots, a_k \in \mathbb{R}$ not all 0 such that $\sum_{i=1}^{k} a_i \mathbf{v}_i = \mathbf{0}$ and $\sum_{i=1}^{k} a_i = 0$. Otherwise, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are called *affine independent*.

Geometrically, some examples of simplices are points, lines and triangles. Formally, the definition requires the notion of affine independence.

Definition 14 (Simplex)**.** A *simplex* σ is the convex hull of a finite set A of affine independent vectors in $\mathbb{R}^n$. The points in A are called the *vertices* of σ and denoted by $V(\sigma)$. The dimension of σ is equal to $|A| - 1$. Namely, a k dimensional simplex, called k-simplex for short, has $k+1$ vertices. The convex hull of an arbitrary subset of the vertices of σ is called a *face* of σ. A proper face of σ is called *facet*.

From the above definitions, it holds that every face is itself a simplex. For simplicity, we denote a simplex as the set of its vertices.

A.1 Simplicial Complexes, Value & Index Functions and Triangulations

Very central to our paper is the notion of *geometric simplicial complexes*, which are used to describe subspaces of $\mathbb{R}^d$. These subspaces consist of simple building blocks, such as points, line segments, triangles, tetrahedra, that are pasted together.

Definition 15 (Simplicial Complex)**.** A *simplicial complex* K is a non-empty set of simplices that satisfies the following properties:

- Each face of a simplex $\sigma \in K$ is also a simplex in K.
- The intersection $\sigma_1 \cap \sigma_2$ of any two simplices $\sigma_1, \sigma_2 \in K$ is a face of both σ_1 and σ_2.

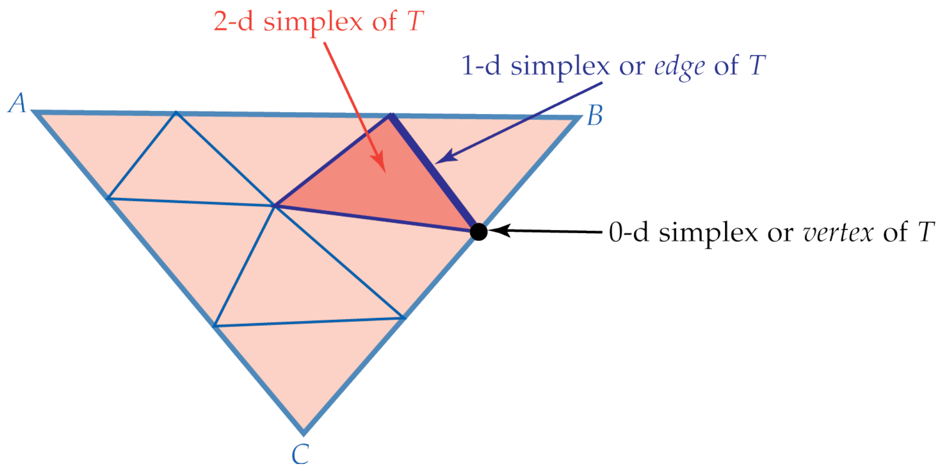

Figure 7: A simplicial complex T that defines a triangulation of the triangle $A - B - C$.

The union of the simplices in K is called the polyhedron of K and is denoted by $\|K\|$. The dimension of K is $\mathsf{dim(K)} := \mathsf{max}_{\sigma \in K}\{\mathsf{dim}(\sigma)\}$ and the vertex set of K, denoted by $V(K)$, is the union of the vertex sets of all its simplices.

We denote by Σ_σ for a simplex σ the simplicial complex that contains all simplices τ such that $\tau \subseteq \sigma$.

According to the above definition, zero-dimensional simplicial complexes correspond to points and one-dimensional simplicial complexes to sets of non-intersecting line segments as shown in Figure 7.

The notion of triangulation relates simplicial complexes with topological spaces.

Definition 16 (Triangulation). A simplicial complex K is a triangulation of a topological space X if $\|K\| \cong X$.

For instance, the boundary of the n-simplex σ_n, namely a simplicial complex containing all proper faces of σ_n, is a triangulation of the sphere S^{n-1}.

Triangulation is a very powerful tool in studying the computational complexity of topological problems, because they allow us to partition a simplicial complex into smaller simplices that are connected in useful ways. We will mainly use the *Kuhn triangulation*, which is described in more detail below. We refer the interested reader to [Matoušek, 2008] and [Munkres, 1984] for further information.

We define two functions of a triangulation, index and value, that are essential for the definition of our computational problems and our reductions.

Definition 17 (Value & Index Functions). Let K be a simplicial complex consisting of k simplices, including the non-full dimensional ones and let $M > k$. We define the *value function* $\mathsf{value} : [M] \to \bar{K}$, where $\bar{K} = K \cup \{\emptyset\}$, to be an efficiently computable function such that

1. value is bijective on K,

2. if $\mathsf{value}(x) = \{v_1, \ldots, v_\ell\}$, then x is called the index of the simplex $\sigma \in K$ with vertices $\{v_1, \ldots, v_\ell\}$, and

3. if $\text{value}(x) = \emptyset$ then x does not correspond to a valid index of any non-empty simplex in K,

We also define the *index function* $\text{index} : \mathbb{R}^n \to [M]$ to be an efficiently computable function such that if $\boldsymbol{x} \in \|K\|$ then $\boldsymbol{x} \in \|\text{value}(\text{index}(\boldsymbol{x}))\|$.

Intuitively, value provides a way to enumerate over the simplices and index on input a point $\mathbf{x}$ returns the simplex that contains $\mathbf{x}$.

Definition 18 (KUHN'S TRIANGULATION [Kuhn, 1960])**.** Kuhn's triangulation is a standard way to triangulate a domain that is a cube. For any $n \in \mathbb{N}$, the cube $[0,1]^n$ is triangulated with granularity $m \in \mathbb{N}$ as follows:

1. the set of vertices of the triangulation is U_m^n, where $U_m = \{0, 1/m, 2/m, \dots, m/m\}$
2. every $x \in (U_m \setminus \{1\})^n$ is the base of the cube containing all vertices $\{y : y_i \in \{x_i, x_i + 1/m\}\}$
3. every such cube is subdivided into $n!$ n-dimensional simplices as follows: for every permutation π of $[n]$, $\sigma = \{y^0, y^1, \dots, y^n\}$ is a simplex, where $y^0 = x$ and $y^i = y^{i-1} + \frac{1}{m} e_{\pi(i)}$ for all $i \in [n]$ (e_i is the ith unit vector)

It is easy to see that Kuhn's triangulation has the following properties:

- For any simplex $\sigma = \{z^1, \dots, z^k\}$ it holds that $\|z^i - z^j\|_\infty \leq 1/m$ for all i, j, and there exists a permutation π of $[k]$ such that $z^{\pi(1)} \leq \cdots \leq z^{\pi(k)}$ (component-wise).
- The restriction of Kuhn's triangulation of $[0,1]^n$ on some subspace $A_1 \times A_2 \times \cdots \times A_n$ of $[0,1]^n$, where for each $i \in [n]$, $A_i \in \{\{0\}, [0,1]\}$, coincides with Kuhn's triangulation of that subspace.
- Every n-dimensional simplex can be indexed by its smallest vertex (component-wise), which is also the base of the cube containing the simplex, and by the permutation that yields this simplex within this cube. Given some index, it is easy to check whether this is a valid index, and if so, output the vertices of the simplex. Thus, the index function can be computed efficiently.
- Given a point $x \in [0,1]^n$, we can efficiently determine the index of a simplex that contains it as follows. First find the base y of a cube of U_m^n that contains x. Next, find a permutation π such that $x_{\pi(1)} - y_{\pi(1)} \geq \cdots \geq x_{\pi(n)} - y_{\pi(n)}$. Then, it follows that (y, π) is the index of a simplex containing x. Thus, the value function can be computed efficiently.
- Given an n-simplex $\{z^0, \dots, z^n\}$ and $i \in \{0, 1, \dots, n\}$, we can efficiently compute the index of the other n-simplex that also has $\{z^0, \dots, z^n\} \setminus \{z_i\}$ as a facet. This is called a *pivot* operation.

A.2 The Borsuk-Ulam Theorem and Tucker's Lemma

Here, we provide the definitions of the problems that we generalize. Note that in Section 3, we explain how the Borsuk-Ulam Theorem can be interpreted under a more general definition, which explains how our definition of Polygon Borsuk-Ulam is indeed a generalization. We start with the Borsuk-Ulam Theorem, which is usually stated as "for every continuous function from S^n to $\mathbb{R}^n$, there exists a point $\mathbf{x} \in \mathbb{R}^n$, such that $f(\mathbf{x}) = f(-\mathbf{x})$". We present an alternative definition (as stated in [Matoušek, 2008]), which is more appropriate for our results.

Theorem A.1 (Borsuk-Ulam Theorem [Borsuk, 1933, Matoušek, 2008])**.** *For every antipodal mapping $f : S^n \to \mathbb{R}^n$ (i.e., a function f which is continuous and $f(\mathbf{x}) = f(-\mathbf{x})$), there exists a point $\mathbf{x} \in S^n$ such that $f(\mathbf{x}) = 0$.*

Tucker's lemma is a well-known combinatorial existence theorem, which is an analogue of the Borsuk-Ulam Theorem. It is usually stated on the d-dimensional unit ball or sphere. For computational purposes the following version is more commonly used.

Theorem A.2 (Tucker's lemma [Tucker, 1945])**.** *Let $m, d \geq 1$. Let $\lambda : ([-m, m] \cap \mathbb{N})^d \to \{\pm 1, \pm 2, \ldots, \pm d\}$ be any labeling that satisfies $\lambda(-x) = -\lambda(x)$ for all $x \in ([-m, m] \cap \mathbb{N})^d$ such that $\exists i$ with $|x_i| = m$. Then there exist two points $x, y \in ([-m, m] \cap \mathbb{N})^d$ with $\|x - y\|_\infty \leq 1$ that have opposite labels, i.e., $\lambda(x) = -\lambda(y)$.*

The corresponding computational problem Tucker is known to be PPA-complete [Papadimitriou, 1994, Aisenberg et al., 2020], even if the dimension is fixed to be $d = 2$.

A.3 $\mathbb{Z}_p$-equivariant tie-breaking

For some of our constructions, we will require a tie-breaking that is $\mathbb{Z}_p$-equivariant and efficiently computable. Observe that non-efficient tie breaking rules exist by carefully choosing one representative for any equivalence class but this is not sufficient for our proofs. We define an efficiently computable rule below for any set $S \in 2^{\mathbb{Z}_p} \setminus \{\emptyset, \mathbb{Z}_p\}$, let $S + i := \{x + i : x \in S\}$.

Definition 19 ($\mathbb{Z}_p$-equivariant tie-breaking)**.** For any prime p, the $\mathbb{Z}_p$-equivariant tie-breaking function $T_p : 2^{\mathbb{Z}_p} \setminus \{\emptyset, \mathbb{Z}_p\} \to \mathbb{Z}_p$ is computed as follows on input $S \in 2^{\mathbb{Z}_p} \setminus \{\emptyset, \mathbb{Z}_p\}$:

1. For every $i \in \mathbb{Z}_p$, write $S + i$ as a bit-string of length p. Namely, construct the bit-string $b(i)$ where the jth bit from the left indicates whether $j \in S + i$ (for $j = 0, \ldots, p - 1$).
2. Output $-i^* \in \mathbb{Z}_p$, where $i^* = \operatorname{argmax}_i b(i)$ ($b(i)$ interpreted as a number in binary).

Lemma A.3. *For any prime p, the $\mathbb{Z}_p$-equivariant tie-breaking function $T_p : 2^{\mathbb{Z}_p} \setminus \{\emptyset, \mathbb{Z}_p\} \to \mathbb{Z}_p$ is well-defined and satisfies for any $S \in 2^{\mathbb{Z}_p} \setminus \{\emptyset, \mathbb{Z}_p\}$:*

- $T_p(S) \in S$
- $T_p(S + i) = T_p(S) + i$ *for all* $i \in \mathbb{Z}_p$.

Proof. If p is prime and $S + i = S$ for some $i \in \mathbb{Z}_p \setminus \{0\}$, then $S \in \{\emptyset, \mathbb{Z}_p\}$. This follows from observing that the corresponding bit strings $b(0)$ and $b(i)$ must be equal and that this implies that the bits of $b(0)$ with index $\{k \cdot i\}_{k \in \mathbb{Z}_p}$ are all equal. Since p is prime, $\{k \cdot i\}_{k \in \mathbb{Z}_p} = \mathbb{Z}_p$.

Hence, $|\{S + i : i \in \mathbb{Z}_p\}| = p$ for any $S \in 2^{\mathbb{Z}_p} \setminus \{\emptyset, \mathbb{Z}_p\}$. Thus, the bit-strings $b(i)$ are all distinct, $T_p(S)$ is unique and T_p is well-defined. Next, by construction, it is easy to see that $T_p(S + i) = -(i^* - i) = T_p(S) + i$. Finally, since $S \neq \emptyset$, i^* will be such that $b(i^*)$ has a 1 in the left-most position. Thus, $0 \in S + i^*$, which implies that $-i^* \in S$. $\square$

Example. Let $p = 3$ and $S = \{2\}$. Then, $S + 0 = \{2\}$, $S + 1 = \{0\}$, $S + 2 = \{1\}$, and $b(0) = 010$, $b(1) = 001$ and $b(2) = 100$ (in binary). From Definition 19, $T_3(S) = 2$.

B (p,n)-BSS-Tucker reduces to $(p,n+1)$-BSS-Tucker

Let (p,n)-BSS-Tucker denote the p-BSS-Tucker problem with dimension parameter n. We have the following lemma.

Lemma B.1. *For all $n \geq 1$ and prime $p \geq 2$, (p,n)-BSS-Tucker reduces to $(p,n+1)$-BSS-Tucker.*

Proof. The domain of (p,n)-BSS-Tucker with Kuhn's triangulation can be written as

$$X_n = \{(c^1, \ldots, c^p) \in U_m^{np} | \forall j \in [n], \exists i \in [p] : c_j^i = 0\}.$$

Note that the subset of X_{n+1} corresponding to $c_{n+1}^1 = \cdots = c_{n+1}^p = 0$ can be identified with X_n. Since we use Kuhn's triangulation in both cases, the triangulations "match".

Let λ be an instance of (p,n)-BSS-Tucker. We construct an instance λ' of $(p,n+1)$-BSS-Tucker as follows. For any vertex $(c^1, \ldots, c^p)$, we set

$$\lambda'(c^1, \ldots, c^p) = \begin{cases} \lambda(c^1, \ldots, c^p), & \text{if } c_{n+1}^1 = \cdots = c_{n+1}^p = 0 \\ (k, n+1) \text{ where } k = T_p(\text{argmax}_{i \in [p]} c_{n+1}^i), & \text{otherwise} \end{cases}$$

In the second case, we have used the tie-breaking rule defined in Definition 19. Since this tie-breaking is $\mathbb{Z}_p$-equivariant, it is easy to see that λ' also satisfies the boundary conditions.

Consider any solution to this instance, i.e., a $(p-1)$-simplex σ that has all labels $(1,\ell), \ldots, (p,\ell)$ for some ℓ. If $\ell = n+1$, then there exists $i \in [p]$ such that $c_{n+1}^i = 0$ *for all* vertices of σ. This follows from the fact that the triangulation is "nice". But then, σ cannot have the label $(i, n+1)$. Hence, it must be that $\ell < n+1$ and σ is contained in the region identified with X_n where $\lambda = \lambda'$. Thus, σ also yields a solution to the original instance. □

C $\mathbb{Z}_p$-star-Tucker is in PPA-p, Full Proof

In this section, we provide the full proof of Theorem 5.3. Namely, we show how to reduce $\mathbb{Z}_p$-star-Tucker to Imbalance-mod-p.

Proof. The proof is a generalization of the construction given by Freund and Todd [1981] for Tucker's lemma. The main difficulties in generalizing their approach are:

- For $p = 2$, when a path hits the boundary, there is a corresponding path that also hits the boundary on the antipodal side, and we can join the two endpoints. For $p > 2$, when a path hits the boundary, there are now $p-1$ other corresponding paths that also hit the boundary. We ensure that no solution occurs there by directing all the edges. We show how to direct the edges consistently and efficiently.

- For $p = 2$, the original construction associates a label with each axis of the domain. For $p > 2$, there are more axes than labels, and so a single label must be associated to multiple axes. This creates imbalanced nodes in the graph that are not solutions. We solve this problem by carefully assigning weights to the edges of the graph.

Sub-orthants. Recall that $d = t(p-1)$. Consider the domain $R^d_{p,m}$ with a labeling function λ given by a Boolean circuit. Let T be Kuhn's triangulation of $R^d_{p,m}$ as described earlier. The domain $R^d_{p,m}$ can be subdivided into what we call sub-orthants, which are orthants of coordinate subspaces. Formally, a sub-orthant is a space of the form $A_1 \times A_2 \times \cdots \times A_d$, where $A_\ell = \{*^{i_\ell}j : 0 \leq j \leq m\}$ or $A_\ell = \{0\}$ for $\ell = 1, \ldots, d$.

We associate a label to every axis of $R^d_{p,m}$ as follows. The label $*^i j$ is associated to the $*^i$-axis of the $[(j-1)(p-1)+\ell]$th copy of $R_{p,m}$, for $\ell = 1, 2, \ldots, p-1$. Thus, every label is associated to exactly $p-1$ axes. For any sub-orthant X, let $S(X)$ denote the set of labels associated with the axes that are used by X. For any simplex σ of T, we let $O(\sigma)$ denote the smallest sub-orthant that contains σ.

For any sub-orthant O and $j \in [t]$, let $r_j(O)$ be the number of coordinates in the range $(j-1)(p-1)+1, \ldots, j(p-1)$ that are equal to 0 in O. In particular, we have $\sum_{j=1}^t r_j(O) = t(p-1) - \dim(O)$. Note that if $|S(O)| = \dim(O)$, then $r_j(O) = p - 1 - |\{i \in \mathbb{Z}_p : *^i j \in S(O)\}|$. We abuse notation and denote $r_j(\sigma) := r_j(O(\sigma))$.

Happy simplices. Let $k \in \{0, 1, \ldots, d\}$. A k-dimensional simplex σ of T is *happy*, if

1. $\dim(O(\sigma)) = k$ *(σ is full-dimensional in its sub-orthant)*
2. $|S(O(\sigma))| = \dim(O(\sigma))$ *(the sub-orthant uses ≤ 1 axis associated with each label)*
3. $S(O(\sigma)) \subseteq \lambda(\sigma)$ *(σ carries all the labels associated with its sub-orthant)*

A happy simplex is called *super-happy* if we actually have $S(O(\sigma)) \subsetneq \lambda(\sigma)$. In particular, the 0-dimensional simplex 0^d is super-happy.

Consider a happy simplex σ that has a facet $\tau \subset \partial R^d_{p,m}$ such that $\lambda(\tau) = S(O(\sigma))$. Such a simplex is called a boundary-happy simplex. If σ is such a simplex, then the simplex that has $\theta\tau$ as a facet is also boundary-happy. Thus, we group $\tau, \theta\tau, \ldots, \theta^{p-1}\tau$ together into an equivalence class $[\tau]$. Every such equivalence class has size exactly p. Formally, let B be the set of all simplices $\tau \subset \partial R^d_{p,m}$ such that $\lambda(\tau) = S(O(\tau))$ and $|S(O(\tau))| = \dim(O(\tau))$. We define an equivalence relation on B by $\tau_1 \equiv \tau_2$ if and only if $\exists i \in \mathbb{Z}_p$ such that $\tau_1 = \theta^i \tau_2$.

We construct a graph G. The vertices of G are the happy simplices of T and the equivalence classes $[\tau]$ (for all $\tau \in B$).

Orientation. We now define an orientation for our simplices. Fix an ordering of the labels, e.g., $*^1 1, *^2 1, \ldots, *^p 1, *^1 2, \ldots$. Let σ be any happy k-simplex, $k \geq 1$, and let $x_0 x_1 \ldots x_k$ be any ordering of its vertices. Let $\widehat{x}_0, \widehat{x}_1, \ldots, \widehat{x}_k \in \mathbb{N}^k$ denote the coordinates of $x_0, x_1, \ldots, x_k$ in $O(\sigma)$, where the coordinates are ordered according to the fixed ordering of the labels. Note that there is at most one coordinate associated to each label (because $|S(O(\sigma))| = k$) and thus the coordinate vectors are uniquely determined. Furthermore, note that the coordinates are non-negative. We define the $k \times k$ matrix $M = [\widehat{x}_0 - \widehat{x}_1, \widehat{x}_0 - \widehat{x}_2, \ldots, \widehat{x}_0 - \widehat{x}_k]$, i.e., the ith column is $\widehat{x}_0 - \widehat{x}_i$. Then, we define the orientation of happy simplex σ with ordering $x_0 x_1 \ldots x_k$ as $\text{or}(\sigma | x_0 \ldots x_k) = \det M$. Note that by construction of the triangulation T, we always have $\det M \in \{-1, +1\}$. Indeed, it is easy to check that M can be transformed into the identity matrix by elementary operations that can only change the sign of the determinant.

Edges. Let σ be a happy k-simplex, $k \geq 1$. If σ is super-happy, then it has a single facet τ such that $\lambda(\tau) = S(O(\sigma))$. If it is not super-happy, then it has exactly two facets τ_1 and τ_2 such that $\lambda(\tau_i) = S(O(\sigma))$, $i = 1, 2$. In any case, any such facet τ of σ yields an edge as follows:

- If τ does not lie in the boundary of the sub-orthant $O(\sigma)$, then there is exactly one other k-simplex σ' in $O(\sigma)$ that also has τ as its facet. σ' is also happy, and we put an edge between σ and σ'.
- If τ lies in the boundary of the sub-orthant $O(\sigma)$, there are two cases:
  - τ lies in $\partial R^d_{p,m}$. In that case, σ is a boundary-happy simplex and we put an edge between σ and $[\tau]$.
  - τ does not lie in $\partial R^d_{p,m}$. Then, τ is a super-happy $(k-1)$-simplex and we put an edge between σ and τ.

In all of these cases, the direction of the edge is determined as follows. Let $x_0x_1\dots x_k$ be the ordering of the vertices of σ such that $\tau = \{x_1, \dots, x_k\}$ and $x_1 \dots x_k$ are ordered according to their labels. If $\mathrm{or}(\sigma|x_0 \dots x_k) = 1$, then the edge is incoming into σ. Otherwise, it is outgoing out of σ. Finally, the weight of the edge is always $\prod_{j=1}^t r_j(\sigma)!$.

By the definition, it follows that there are three types of edges:

- (Type 1) An edge between two happy k-simplices σ_1, σ_2 that lie in the same sub-orthant and share a facet τ with $\lambda(\tau) = S(O(\sigma_i))$, $i = 1, 2$.
- (Type 2) An edge between a happy simplex σ and its super-happy facet τ such that $\lambda(\tau) = S(O(\sigma))$.
- (Type 3) An edge between a boundary-happy simplex σ and its facet equivalence class $[\tau]$.

An edge of Type 2 or 3 is always "created" by exactly one of its endpoints. Thus, its direction and weight are well-defined. An edge of Type 1 however, is "created" by both of its endpoints and we will prove that it is well-defined, i.e., both endpoints agree on its direction and weight. For the weight this is easy to see, since $O(\sigma_1) = O(\sigma_2)$ implies that $r_j(\sigma_1) = r_j(\sigma_2)$ for all j. For the direction, we postpone this consistency check to the end of the proof.

We now prove that all vertices of G that do not yield a solution are balanced modulo p, except 0^d.

The trivial solution 0^d. We have $\lambda(0^d) = *^i j$. There are exactly $p-1$ sub-orthants O such that $S(O) = \{*^i j\}$ and each of them contains a happy 1-simplex σ that has 0^d as a facet. It follows that 0^d has $p-1$ edges, each with weight $((p-1)!)^t/(p-1)$. Furthermore, all the edges are outgoing, because $\mathrm{or}(\sigma|x_0 0^d) = 1$ (i.e., incoming into σ). It follows that the total imbalance of 0^d is $(p-1)((p-1)!)^t/(p-1) = ((p-1)!)^t = (-1)^t \mod p$, where we used $(p-1)! = -1 \mod p$ since p is prime (Wilson's theorem). It follows that 0^d is always a valid trivial solution for Imbalance-mod-p, because $(-1)^t \neq 0 \mod p$ for all $t \geq 1$.

Happy, but not super-happy. Consider a happy k-simplex σ with two facets τ_1, τ_2 that satisfy $\lambda(\tau_i) = S(O(\sigma))$, $i = 1, 2$. Then, σ has two edges and they both have the same weight. Let $x_0x_1 \dots x_k$ be the ordering of σ such that $\tau_1 = \{x_1, \dots, x_k\}$ and $x_1 \dots x_k$ are ordered according to their labels. Let $i \in [k]$ be the index such that x_i and x_0 have the same label. In particular, $\tau_2 = \{x_1, \dots, x_{i-1}, x_0, x_{i+1}, \dots, x_k\}$ and $x_1 \dots x_{i-1} x_0 x_{i+1} \dots x_k$ are ordered according to their labels. We have

$$\begin{aligned}\det[\widehat{x}_0 - \widehat{x}_1, \widehat{x}_0 - \widehat{x}_2, \dots, \widehat{x}_0 - \widehat{x}_k] &= \det[\widehat{x}_i - \widehat{x}_1, \dots, \widehat{x}_i - \widehat{x}_{i-1}, \widehat{x}_0 - \widehat{x}_i, \widehat{x}_i - \widehat{x}_{i+1}, \dots, \widehat{x}_i - \widehat{x}_k] \\ &= -\det[\widehat{x}_i - \widehat{x}_1, \dots, \widehat{x}_i - \widehat{x}_{i-1}, \widehat{x}_i - \widehat{x}_0, \widehat{x}_i - \widehat{x}_{i+1}, \dots, \widehat{x}_i - \widehat{x}_k]\end{aligned}$$

where we first subtracted the ith column from all other columns, and then multiplied the ith column by -1. It follows that $\mathrm{or}(\sigma|x_0 \dots x_k) = -\mathrm{or}(\sigma|x_i x_1 \dots x_{i-1} x_0 x_{i+1} \dots x_k)$. Thus, one edge is incoming and the other outgoing, i.e., σ is balanced.

Equivalence class. Consider an equivalence class $[\tau]$. Let σ be the happy k-simplex that has τ as a facet. In G, $[\tau]$ has exactly p edges: one with each of $\sigma, \theta\sigma, \theta^2\sigma, \dots, \theta^{p-1}\sigma$. Since $S(O(\theta^i\sigma)) = \theta^i S(O(\sigma))$ for all i, it follows that $r_j(\theta^i\sigma) = r_j(\sigma)$ for all i, j. Thus, all p edges have the same weight. Let $x_0 \dots x_k$ be the ordering of σ such that $\tau = \{x_1, \dots, x_k\}$ and $x_1 \dots x_k$ are ordered according to their labels. Let $y_i = \theta x_i$ for all i. Then, $y_1 \dots y_k$ might not be ordered according to their labels. We let π denote the permutation that we would have to apply to order them correctly. As before, $\widehat{x}_i$ denotes the coordinates of x_i restricted to $O(\sigma)$, where the coordinates are ordered according to the associated label. $\widehat{y}_i$ denotes the coordinates of y_i restricted to $O(\theta\sigma)$, where the coordinates are ordered according to the associated label. Since the associated labels have changed according to θ, it follows that if we re-order the coordinates of $\widehat{x}_i$ according to π we obtain $\widehat{y}_i$ for all $i = 0, 1, \dots, k$. Thus, we have

$$\begin{aligned}
\mathrm{or}(\theta\sigma|y_0\pi(y_1 \dots y_k)) &= \mathrm{sgn}(\pi)\det[\widehat{y}_0 - \widehat{y}_1, \widehat{y}_0 - \widehat{y}_2, \dots, \widehat{y}_0 - \widehat{y}_k] \\
&= \mathrm{sgn}(\pi)^2[\widehat{x}_0 - \widehat{x}_1, \widehat{x}_0 - \widehat{x}_2, \dots, \widehat{x}_0 - \widehat{x}_k] \\
&= \mathrm{or}(\sigma|x_0 \dots x_k)
\end{aligned}$$

It follows that all edges of $[\tau]$ are directed the same way, i.e., they are all incoming or all outgoing. Since there are p edges and they also have the same weight, it follows that $[\tau]$ has imbalance 0 modulo p. In this argument we assumed that λ satisfies the boundary conditions. Thus, if $[\tau]$ is not balanced modulo p, we obtain a counter-example, which is a solution.

Super-happy. Consider a super-happy k-simplex σ, $k \geq 1$. Note that σ has a single facet τ that satisfies $\lambda(\tau) = S(O(\sigma))$. Thus, σ "creates" a single edge. Let $x_0 \dots x_k$ be the ordering of σ such that $\tau = \{x_1, \dots, x_k\}$ and $x_1 \dots x_k$ are ordered according to their labels. The edge has weight $\prod_{j=1}^t r_j(\sigma)!$ and it is incoming if $\mathrm{or}(\sigma|x_0 \dots x_k) = 1$, outgoing otherwise.

Since σ is super-happy, we have $\lambda(x_0) \notin \lambda(\tau) = S(O(\sigma))$. Let $*^i\ell = \lambda(x_0)$. If $\{*^j\ell | j \in [p] \setminus \{i\}\} \subseteq S(O(\sigma))$, or equivalently if $r_\ell(\sigma) = 0$, then σ yields a solution. Otherwise, there are exactly $r_\ell(\sigma)$ different sub-orthants O such that $O(\sigma) \subset O$ and $S(O) = S(O(\sigma)) \cup \{*^i\ell\}$. Thus, there are exactly $r_\ell(\sigma)$ happy $(k+1)$-simplices ρ such that σ is a facet of ρ and ρ is happy because of σ (i.e., $S(O(\rho)) = \lambda(\sigma)$). It follows that σ has $r_\ell(\sigma)$ additional edges (apart from the one it "created"). Each of these edges has weight $\prod_{j=1}^t r_j(\rho)! = (r_\ell(\sigma))^{-1}\prod_{j=1}^t r_j(\sigma)!$, since $r_\ell(\rho) = r_\ell(\sigma) - 1$. Thus, if these $r_\ell(\sigma)$ edges have opposite direction to the edge "created" by σ (from the perspective of σ), σ will be balanced.

Consider any such ρ. Let $x_0 \dots x_{k+1}$ be the ordering of ρ such that $\sigma = \{x_1, \dots, x_{k+1}\}$ and $x_1 \dots x_{k+1}$ are ordered according to their labels. Let $t \in [k+1]$ be the index of label $*^i\ell$ if we order the labels in $S(O(\rho))$. Then, we also have that $\tau = \sigma \setminus \{x_t\}$. Furthermore, $x_1 \dots x_{t-1}x_{t+1} \dots x_{k+1}$ are also ordered according to their labels. From the perspective of σ, the edge it "created" is directed according to $\mathrm{or}(\sigma|x_t x_1 \dots x_{t-1} x_{t+1} \dots x_{k+1})$ and the edge created by ρ is directed according to $-\mathrm{or}(\rho|x_0 x_1 \dots x_{k+1})$. As before, let $\widehat{x}_j$ denote the coordinates of x_j restricted to $O(\rho)$, where the coordinates are ordered according to the associated label. Let $\bar{x}_j$ denote the coordinates of x_j restricted to $O(\sigma)$, where the coordinates are ordered according to the associated label. Note that if we remove the tth coordinate from $\widehat{x}_j$, then we obtain $\bar{x}_j$. We now have

$$\begin{aligned}
\mathrm{or}(\rho|x_0x_1\dots x_{k+1}) &= \det[\widehat{x}_0 - \widehat{x}_1, \dots, \widehat{x}_0 - \widehat{x}_{k+1}] \\
&= \det[\widehat{x}_t - \widehat{x}_1, \dots, \widehat{x}_t - \widehat{x}_{t-1}, \widehat{x}_0 - \widehat{x}_t, \widehat{x}_t - \widehat{x}_{t+1}, \dots, \widehat{x}_t - \widehat{x}_{k+1}] \\
&= (-1)^{t+t}\det[\bar{x}_t - \bar{x}_1, \dots, \bar{x}_t - \bar{x}_{t-1}, \bar{x}_t - \bar{x}_{t+1}, \dots, \bar{x}_t - \bar{x}_{k+1}] \\
&= \mathrm{or}(\sigma|x_tx_1\dots x_{t-1}x_{t+1}\dots x_{k+1})
\end{aligned}$$

where we first subtracted the tth column from all other columns, and then we used Laplace's determinant formula along the tth row. Note that the tth entry in $\widehat{x}_t - \widehat{x}_j$ is 0 for all $j \in [k+1]$ and it is 1 in $\widehat{x}_0 - \widehat{x}_t$.

Consistency. The only thing that remains to be checked is that edges of Type 1 are well-defined, in terms of the direction. Let σ_1, σ_2 be two happy k-simplices that lie in the same sub-orthant and share a facet τ with $\lambda(\tau) = S(O(\sigma_i))$, $i = 1, 2$. Let $\{x_1, \dots, x_k\} = \tau$ and $x_1 \dots x_k$ be the ordering according to their labels. Let $\{x_0\} = \sigma_1 \setminus \tau$ and $\{x_0'\} = \sigma_2 \setminus \tau$. We want to show that $\mathrm{or}(\sigma_1|x_0x_1\dots x_k)$ and $\mathrm{or}(\sigma_2|x_0'x_1\dots x_k)$ have opposite signs. This can be proved directly combinatorially by using the way the triangulation is constructed, but we provide a proof that is more general here. Let $\phi : \mathbb{R}^k \to \mathbb{R}^k$ be the unique linear function such that $\phi(\widehat{x}_0 - \widehat{x}_1) = \widehat{x}_0' - \widehat{x}_1$ and $\phi(\widehat{x}_1 - \widehat{x}_i) = \widehat{x}_1 - \widehat{x}_i$ for all $i \in [k] \setminus \{1\}$. ϕ is unique, because $\widehat{x}_0 - \widehat{x}_1, \widehat{x}_1 - \widehat{x}_2, \dots, \widehat{x}_1 - \widehat{x}_k$ form a basis of $\mathbb{R}^k$. ϕ is the identity function on the hyperplane given by $\widehat{x}_1 - \widehat{x}_2, \dots, \widehat{x}_1 - \widehat{x}_k$ and maps $\widehat{x}_0 - \widehat{x}_1$ to $\widehat{x}_0' - \widehat{x}_1$. Since x_0 and x_0' lie on opposite sides of the hyperplane defined by τ, it follows that $\widehat{x}_0 - \widehat{x}_1$ and $\widehat{x}_0' - \widehat{x}_1$ lie on opposite sides of the hyperplane on which ϕ is the identity. It follows that $\det\phi < 0$. Thus, we can write

$$\begin{aligned}
\det[\widehat{x}_0' - \widehat{x}_1, \dots, \widehat{x}_0' - \widehat{x}_k] &= \det[\widehat{x}_0' - \widehat{x}_1, \widehat{x}_1 - \widehat{x}_2 \dots, \widehat{x}_1 - \widehat{x}_k] \\
&= \det\phi \det[\widehat{x}_0 - \widehat{x}_1, \widehat{x}_1 - \widehat{x}_2 \dots, \widehat{x}_1 - \widehat{x}_k] \\
&= \det\phi \det[\widehat{x}_0 - \widehat{x}_1, \dots, \widehat{x}_0 - \widehat{x}_k]
\end{aligned}$$

and the claim follows. $\square$